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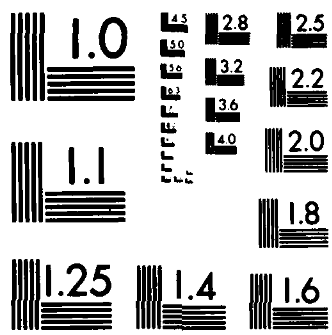

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ROBUST LINEAR FILTERING FOR MULTIVARIABLE STATIONARY TIME SERIES

Haralampos Tsaknakis and P. Papantoni-Kazakos
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Technical Report TR-83-6

April 1983

Department of

Electrical Engineering and Computer Science

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Abstract

The problem of asymptotic, non-causal linear filtering for statistically contaminated multivariable stationary time series is considered. The spectra of both the signal and the noise components of the observation process are assumed to belong to certain convex and compact classes. The minimax criterion of optimality is adopted, and for some specific spectral classes the corresponding solutions are found. The performance of those solutions is studied, where the performance criteria used are efficiency, error variation within the classes, and breakdown curves or points. Some examples are studied quantitively.

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1. Introduction

We consider non-causal filtering for stationary information processes embedded in additive noise. The problem and its solution are well established when both the information and noise processes are statistically well-known, stationary, and mutually uncorrelated. The reader may refer to the work by Kolmogorov [4], and to the books by Wiener [9] and Hannan [2]. Let us now assume that the information and noise processes are statistically contaminated. Then, a single non-causal filter is sought that will provide satisfactory performance for every information process-noise process pair, within the statistically contaminated classes. If satisfactory performance implies qualitative robustness, then a nonlinear operation on the data of the information process should be in general imposed, before transmission through the noise channel. Such a stationary nonlinear operation maps a compact class of stationary processes onto another compact class of stationary processes. If the nonlinear operation is appropriately designed, a linear filter will maintain the characteristics of qualitative robustness. For qualitative analysis of the above, the interested reader may seek reference [6]. From now on we will assume that a proper nonlinear operation has been adopted, and we will concentrate on the stationary processes induced by this operation and the class of information processes. We will name those induced processes, information processes. We will assume that the noise process lies within another compact class of stationary processes, and that the members of this class and the members of the class of information processes are mutually uncorrelated. Then, we will focus on the design of a robust linear filter, adopting a saddle point game theoretic approach. We point out here that the same approach was adopted in [8], where noiseless robust prediction and interpolation of multivariable stationary processes was considered. Also, considering robust linear filtering for scalar, stationary information and noise processes, the interested reader may seek references [3] and [7].

In the present paper, we consider the robust linear non-causal filtering problem, for multivariable stationary information and noise processes. We adopt asymptotic linear operations, and we formulate the problem as a game with saddle point solution. We find this solution explicitly, when the information and noise processes lie within either one of two compact classes of multivariable stationary processes. One of the compact classes represents linear contamination of a nominal multivariable process. The other class includes multivariable processes with fixed energy on prespecified frequency quantiles.

The organization of the paper is as follows. In section 2, we formulate the problem, and we define the compact classes of multivariable stationary processes. In section 3, we find the robust solution, when the compact classes for the information and the noise processes both represent linear contamination of a nominal multivariable process. In section 4, we find the robust linear filter when one of the compact classes represents linear contamination of a nominal multivariable process, and the other compact class includes processes with fixed energy on prespecified frequency quantiles. In section 5, we find the robust linear filter, when both the compact classes include processes with fixed energy on prespecified frequency quantiles. In section 6, we present some criteria for the performance evaluation of the robust filters in sections 3, 4, and 5, and we use those criteria to study the performance of the robust filters in some specific examples.

2. Problem Formulation

We consider the asymptotic non-causal linear filtering problem for stationary multivariable time series, when the statistical structure of both the information and the noise processes is vaguely or incompletely specified. We assume that the noise process is additive to the information process, and that the two processes are zero mean and mutually uncorrelated. Let $X^n(j)$; $j = \ldots, -1, 0, 1, \ldots$ denote a sequence of n-dimensional data vectors from the multivariable information process. Let $Y^n(j)$; $j = \ldots, -1, 0, 1, \ldots$ denote such a sequence from the noise process. Then, the observation data sequence $Z^n(j)$; $= \ldots, -1, 0, 1, \ldots$ is such that $Z^n(j) = X^n(j) + Y^n(j)$; $\forall\, j$, and the noncausal linear filter performs the operation $\sum_{k=-\infty}^{\infty} A_k^n Z^n(k)$ to extract the vector $X^n(0)$; where $\{A_k^n\}$ is some sequence of constant nxn matrices (whose properties will be stated later), such that in the frequency domain the matrix polynomial $\underline{H}_n(\omega) \triangleq \sum_{k=-\infty}^{\infty} A_k^n e^{jk\omega}$ exists, and it describes the linear filter uniquely. In the parametric linear filter problem, the information and noise processes are considered well-known. Then, if the spectral density matrices of both those processes exist (as in [8]), and they are denoted respectively by $\underline{f}_s^n(\omega)$ and $\underline{f}_N^n(\omega)$, they are both Hermitian and nonnegative definite, and their elements are Lebesque integrable functions on the measurable space $([-\pi,\pi], \mathcal{B}_\pi)$; where $\mathcal{B}_\pi$ the Borel field on $[-\pi,\pi]$. If the asumptotic non-causal linear filter signified by $\underline{H}_n(\omega)$ is adopted, and the mean square performance criterion is considered, it is then well-known [2] that the error $e(\underline{f}_s^n, \underline{f}_N^n, \underline{H}_n)$ induced is given by the following expression.

$$e(\underline{f}_s^n,\underline{f}_N^n,\underline{H}_n)=\mathrm{tr}(2\pi)^{-1}\int_{-\pi}^{\pi}\{[I_n-\underline{H}_n(\omega)]\underline{f}_s^n(\omega)[I_n-\underline{H}_n^*(\omega)]^T+\underline{H}_n(\omega)\underline{f}_N^n(\omega)\underline{H}_n^{*T}(\omega)\}\,d\omega \tag{1}$$

; where tr means trace, * and T signify conjugate and transpose respectively, and I_n is the nxn identity matrix.

From expression (1) we observe that the mean square error $e(\underline{f}_s^n, \underline{f}_N^n, \underline{H}_n)$ is only a function of the spectral density matrices of both the information and noise processes. Thus, if we now assume that the statistical structure of the information and noise processes is incompletely specified, we can represent this incompleteness solely by uncertainty in the description of the spectral density matrices $\underline{f}_s^n(\omega)$ and $\underline{f}_N^n(\omega)$. We will formulate our approach below. We will also drop the index n, for simplicity in our notation.

Let the spectral density matrix $\underline{f}_s(\omega)$ of the information process belong to a class C_s. Let the spectral density matrix $\underline{f}_N(\omega)$ of the noise process belong to a class C_N. Let the members of each class be Hermitian, nonnegative definite matrices, defined on the interval $[-\pi,\pi]$, containing no impulses anywhere on $[-\pi,\pi]$, and being nonsingular for all $\omega\varepsilon[-\pi,\pi]$. The no impulses and nonsingularity restrictions do not cause serious loss in generality, and they can be relaxed if necessary. If impulses exist at a finite number of points, they can be approximated by Gaussian functions with arbitrarily small variances. If singularities exist, each can be analyzed separately via lower rank matrices. Let S_f be the class of linear filters whose matrix coefficients $\{A_k\}$ are the Laurent series expansion coefficients of a holomorphic matrix valued function, within an annulus containing the unit circle in the complex plane. Then, each member $\{A_k\}$ in S_f is uniquely described by the matrix polynomial $\underline{H}(\omega) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega}$, in the frequency domain. Let us consider the space S of all matrices defined on $[-\pi,\pi]$. Then, for $A(\omega)\varepsilon S$ and $B(\omega)\varepsilon S$ we define the metric:

$$d(A,B) \triangleq tr(2\pi)^{-1}\int_{-\pi}^{\pi} [A(\omega)-B(\omega)][A^*(\omega)-B^*(\omega)]^T d\omega \qquad (2)$$

The class S_f of linear filters is convex and locally compact with respect to the metric in (2). We will consider classes C_s and C_N that are also convex and locally compact with respect to the metric in (2). Consider now a game on $(C_s x C_N) x S_f$, with payoff function $e(\underline{f}_s, \underline{f}_N, \underline{H})$; where the latter is the error expression in (1).

The game has a saddle-point solution $(\underline{f}_s^o, \underline{f}_N^o, \underline{H}^o)$, where $\underline{f}_s^o \varepsilon C_s$, $\underline{f}_N^o \varepsilon C_N$, and $H^o \varepsilon S_f$, iff:

$$e(\underline{f}_s,\underline{f}_N,\underline{H}^o) \leq e(\underline{f}_s^o,\underline{f}_N^o,\underline{H}^o) \leq e(\underline{f}_s^o,\underline{f}_N^o,\underline{H}) \;\; ; \forall\ \underline{H}\varepsilon S_f,\ \forall(\underline{f}_s,\underline{f}_N)\ \varepsilon\ C_s x C_N \tag{3}$$

If a saddle point solution exists, we will name $\underline{H}^o$ the robust filter. But the error function $e(\underline{f}_s, \underline{f}_N, \underline{H})$ in (1) is clearly linear with respect to $\underline{f}_s$ and $\underline{f}_N$, and it is convex with respect to $\underline{H}$. Then, a saddle point solution always exists [5]. Furthermore, if $(\underline{f}_s^o, \underline{f}_N^o, \underline{H}^o)$ is such a solution, then:

$$e(\underline{f}_s^o, \underline{f}_N^o, \underline{H}^o) = \inf_{\underline{H}\varepsilon S_f} \sup_{(\underline{f}_s,\underline{f}_N)\varepsilon C_s x C_N} e(\underline{f}_s, \underline{f}_N, \underline{H}) =$$

$$= \sup_{(\underline{f}_s,\underline{f}_N)\varepsilon C_s x C_N} \inf_{\underline{H}\varepsilon S_f} e(\underline{f}_s, \underline{f}_N, \underline{H}) \tag{4}$$

Let us define.

$$e_m(\underline{f}_s, \underline{f}_N) \triangleq \inf_{\underline{H}\varepsilon S_f} e(\underline{f}_s, \underline{f}_N, \underline{H}) \tag{5}$$

Then, we have directly from [2].

$$e_m(\underline{f}_s,\underline{f}_N) = tr(2\pi)^{-1}\int_{-\pi}^{\pi} \underline{f}_s(\omega)\,[\underline{f}_s(\omega)+\underline{f}_N(\omega)]^{-1}\,\underline{f}_N(\omega)\,d\omega = tr(2\pi)^{-1}\int_{-\pi}^{\pi} [\underline{f}_s^{-1}(\omega)+\underline{f}_N^{-1}(\omega)]^{-1} d\omega \tag{6}$$

attained for $\underline{H}(\omega) = \underline{f}_s(\omega)[\underline{f}_s(\omega)+\underline{f}_N(\omega)]^{-1}$; for almost all ω in $[-\pi,\pi]$. Thus, for the solution of the saddle-point game, it is equivalent to search for the supremum of the expression $e_m(\underline{f}_s, \underline{f}_N)$ on C_s x C_N. It is easily observed that $e_m(\underline{f}_s, \underline{f}_N)$ is concave with respect to both $\underline{f}_s$ and $\underline{f}_N$.

As in [8], we will consider the following two classes of spectral density matrices that are both convex and locally compact with respect to the metric in (2):

$$\mathcal{F}_{L,\varepsilon} = \{\underline{f}(\omega):\ \underline{f}(\omega) = (1-\varepsilon)\underline{f}_o(\omega) + \varepsilon\ \underline{h}(\omega)\ ;\ \omega \in [-\pi,\pi]$$

; where ε given and such that : $0 < \varepsilon < 1$

$\underline{f}_o(\omega)$ well-known positive definite Hermitian matrix

$\underline{h}(\omega)$ nonnegative definite Hermitian matrix satisfying the energy

constraint: $(2\pi)^{-1}\ \mathrm{tr}\int_{-\pi}^{\pi} \underline{h}(\omega)d\omega \leq W$; for given W}

$$\mathcal{F}_Q = \{\underline{f}(\omega):\ \mathrm{tr}\int_{A_i} \underline{f}(\omega)d\omega = c_i\ ;\ i = 1,\ldots k,\ \mathrm{tr}\int_{-\pi}^{\pi} \underline{f}(\omega)d\omega = c$$

; where $\{A_i\}$ measurable disjoint subsets of $[-\pi,\pi]$,

$\bigcup_{i=1}^{k} A_i \subset [-\pi,\pi]$, $c > \sum_{i=1}^{k} c_i$, $\underline{f}(\omega)$ positive definite Hermitian matrix}

Class $\mathcal{F}_{L,\varepsilon}$ represents linear ε-contamination of a nominal spectral density matrix $\underline{f}_0(\omega)$. Class $\mathcal{F}_Q$ includes spectral density matrices whose energy is fixed within prespecified frequency quantiles. Both classes satisfy the necessary topological properties, for the existence of a solution for the game in (3). In the subsequent sections, we will find the solutions of the game, for the following three cases:

i. $\mathcal{C}_s = \mathcal{F}_{L,\varepsilon_s}$ and $\mathcal{C}_N = \mathcal{F}_{L,\varepsilon_N}$

ii. $\mathcal{C}_s = \mathcal{F}_{L,\varepsilon}$ and $\mathcal{C}_N = \mathcal{F}_Q$, and conversely

iii. $\mathcal{C}_s = \mathcal{F}_Q$ and $\mathcal{C}_N = \mathcal{F}_Q$

3. The Game Solution on $\mathcal{F}_{L,\varepsilon} \times \mathcal{F}_{L,\varepsilon}$

In the present section, we consider the case where $\mathcal{C}_s = \mathcal{F}_{L,\varepsilon_s}$; with energy constraint W_s, and $\mathcal{C}_N = \mathcal{F}_{L,\varepsilon_N}$; with energy constraint W_N. As we saw in section 2, both the $\mathcal{C}_s$ and $\mathcal{C}_N$ classes are then convex and locally compact with respect to the metric in (2), and then the game reduces to obtaining the supremum of the expression $e_m(\underline{f}_s, \underline{f}_N)$ in (6), on $\mathcal{F}_{L,\varepsilon_s} \times \mathcal{F}_{L,\varepsilon_N}$. Let $\underline{f}_{os}(\omega)$ and $\underline{f}_{oN}(\omega)$ be the nominal spectral densities in the classes $\mathcal{F}_{L,\varepsilon_s}$ and $\mathcal{F}_{L,\varepsilon_N}$ respectively. Let,

$$(2\pi)^{-1}\ \mathrm{tr}\int_{-\pi}^{\pi} \underline{f}_{os}(\omega)\ d\omega = W_{os}$$

$$(2\pi)^{-1}\ \mathrm{tr}\int_{-\pi}^{\pi} \underline{f}_{oN}(\omega)\ d\omega = W_{oN} \tag{7}$$

Then, the solution of the game on $F_{L,\varepsilon_s} \times F_{L,\varepsilon_N}$ reduces to the following optimization problem.

Find the supremum of

$$e_m(\underline{f}_s, \underline{f}_N) \triangleq (2\pi)^{-1}\ \mathrm{tr}\int_{-\pi}^{\pi} [\underline{f}_s^{-1}(\omega)+\underline{f}_N^{-1}(\omega)]^{-1}\ d\omega$$

subject to the constraints:

$$\underline{f}_s(\omega) - (1-\varepsilon_s)\ \underline{f}_{os}(\omega) \geq 0 \tag{3A}$$

$$\underline{f}_N(\omega) - (1-\varepsilon_N)\ \underline{f}_{oN}(\omega) \geq 0$$

$$\mathrm{tr}\int_{-\pi}^{\pi} \underline{f}_s(\omega)\ d\omega \leq 2\pi\ [(1-\varepsilon_s)\ W_{os} + \varepsilon_s\ W_s] \triangleq W_1$$

$$\mathrm{tr}\int_{-\pi}^{\pi} \underline{f}_N(\omega)\ d\omega \leq 2\pi\ [(1-\varepsilon_N)\ W_{oN} + \varepsilon_N\ W_N] \triangleq W_2$$

; where if $B(\omega)$ is a matrix defined on $[-\pi,\pi]$, $B(\omega) \geq 0$ means nonnegative definite for all ω in $[-\pi,\pi]$.

The expression $e_m(\underline{f}_s, \underline{f}_N)$ in the optimization problem in (3A) involves explicitly both the eigenfunctions and the eigenvectors of the spectral densitiy matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$; for all ω in $[-\pi,\pi]$. This induces complications in the optimization process. Those complications can be resolved, however. In particular, we will first derive an

upper bound on $e_m(\underline{f}_s, \underline{f}_N)$, that involves eigenfunctions only. Then, we will find the supremum of this upper bound, and we will show that this supremum is attained for any eigenvectors. To formulate our approach, we will first present a lemma, and two corollaries. We will then apply the lemma and the corollaries to the problem in (3A).

Lemma

Let m be a positive measure defined on a measurable space (Ω, F). Let $A\varepsilon F$, and let f, g be two real functions mapping A onto the real line R. Let f and g be also integrable on A and such that $f+g > 0$, a.e. in A. Then,

$$\int_A \frac{fg}{f+g}\, dm \le \frac{\int_A f\, dm \cdot \int_A g\, dm}{\int_A f\, dm + \int_A g\, dm}$$

with equality if and only if $f = cg$, a.e. in A, for some constant c.

Corollary 1

For any real numbers x_i, y_i ; $i = 1,\ldots,n$, such that $x_i + y_i > 0$; $\forall i$, the following inequality holds,

$$\sum_{i=1}^{n} \frac{x_i\, y_i}{x_i+y_i} \le \frac{\sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i}$$

with equality iff $\dfrac{x_i}{y_i} = \dfrac{x_j}{y_j}$; $\forall\, i \neq j$.

The proof of the lemma is in the appendix. The result in corollary 1 follows from the lemma, the definition of a set $\{A_i\}$ such that $A_i \cap A_j = \emptyset$; $\forall\, i \neq j$, $\bigcup_{j=1}^{n} A_i = A$, and for $f = \sum_{i=1}^{n} x_i\, 1_{A_i}$ and $g = \sum_{i=1}^{n} y_i\, 1_{A_i}$; where 1_{A_i} the indicator function of A_i.

Corollary 2

Let A and B be two nxn Hermitian and positive definite constant matrices. Let $\{\lambda_i(A);\ 1 \leq i \leq n\}$ and $\{\lambda_i(B);\ 1 \leq i \leq n\}$ be the sets of their ordered eigenvalues; that is, $\lambda_i(A) \geq \lambda_{i+1}(A)$ and $\lambda_i(B) \geq \lambda_{i+1}(B)$, for all i. Then, the following inequality holds,

$$\operatorname{tr}\,(A^{-1} + B^{-1})^{-1} \leq \left(\left[\sum_{i=1}^{n} \lambda_i(A)\right]^{-1} + \left[\sum_{i=1}^{n} \lambda_i(B)\right]^{-1}\right)^{-1}$$

; with equality if and only if A = cB, for some scalar constant c.

The proof of corollary 2 is in the appendix, and it evolves from the lemma.

We will now apply the lemma and the two corollaries to the optimization problem (3A). Let us consider the nxn spectral density matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$. Since both the nominal spectral density matrices $\underline{f}_{os}(\omega)$ and $\underline{f}_{oN}(\omega)$ in (3A) have been assumed positive definite; for all ω in $[-\pi,\pi]$, and due to the nonnegative definite constraints in (3A), the spectral density matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$ are positive definite for all ω in $[-\pi,\pi]$. They are also Hermitian. Let us temporarily fix ω, and let us denote by $\{\lambda_{is}(\omega);\ 1 \leq i \leq n\}$ and $\{\lambda_{iN}(\omega);\ 1 \leq i \leq n\}$ the sets of ordered eigenfunctions for the matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$ respectively; where $\lambda_{is}(\omega) \geq \lambda_{i+1,s}(\omega)$ and $\lambda_{iN}(\omega) \geq \lambda_{i+1,N}(\omega); \forall i$ and $\forall\ \omega \varepsilon [-\pi,\pi]$. Then, the matrices $\underline{f}_s^{-1}(\omega)$ and $\underline{f}_N^{-1}(\omega)$ are also Hermitian and positive definite, and their respective sets of ordered eignefunctions are $\{\lambda_{is}^{-1}(\omega);\ 1 \leq i \leq n\}$ and $\{\lambda_{iN}^{-1}(\omega);\ 1 \leq i \leq n\}$; with $\lambda_{i+1,s}^{-1}(\omega) \geq \lambda_{is}^{-1}(\omega)$ and $\lambda_{i+1,N}^{-1}(\omega) \geq \lambda_{iN}^{-1}(\omega)$, for all i and all ω. If we now apply corollary 2 to the spectral density matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$, for some fixed ω, we obtain:

$$\operatorname{tr}\,(\underline{f}_s^{-1}(\omega) + \underline{f}_N^{-1}(\omega))^{-1} \leq \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega)\right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega)\right]^{-1}\right)^{-1} \qquad (8)$$

; with equality iff $\underline{f}_s(\omega) = c(\omega) \cdot \underline{f}_N(\omega)$, for some scalar $c(\omega)$.

Directly from the expression $e_m(\underline{f}_s, \underline{f}_N)$ in (6) and from (8), we now obtain:

$$e_m(\underline{f}_s, \underline{f}_N) \leq (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega) \right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega) \right]^{-1} \right)^{-1} d\omega$$

$$\triangleq e_b(\underline{f}_s, \underline{f}_N) \qquad (9)$$

; where if $\underline{f}_s(\omega) = c(\omega) \cdot \underline{f}_N(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$, for some scalar function $c(\omega)$ defined on $[-\pi,\pi]$, then equality holds above.

We will now transform the optimization problem (3A) into another optimization problem, that involves eigenfunctions only. To do that, we will substitute the objective function $e_m(\underline{f}_s, \underline{f}_N)$ in (3A), by its upper bound $e_b(\underline{f}_s, \underline{f}_N)$ in (9). We will also substitute the constraints in (3A) by constraints that involve eigenfunctions only. We will show that the solution of the transformed problem is a sufficient solution for the problem (3A).

Let $\underline{f}_{os}(\omega)$ and $\underline{f}_{oN}(\omega)$ be the nominal, positive definite spectral density matrices in the classes F_{L,ε_s} and F_{L,ε_N} respectively. Let $\{\lambda^o_{is}(\omega);\ 1 \leq i \leq n\}$ be the set of ordered eigenfunctions of the matrix $\underline{f}_{os}(\omega)$ on $[-\pi,\pi]$. Let $\{\lambda^o_{iN}(\omega);\ 1 \leq i \leq n\}$ be the ordered eigenfunctions of $\underline{f}_{oN}(\omega)$ on $[-\pi,\pi]$; where $\lambda^o_{is}(\omega) \geq \lambda^o_{i+1,s}(\omega)$; $\forall$ i, $\forall\ \omega\varepsilon[-\pi,\pi]$, and $\lambda^o_{iN}(\omega) \geq \lambda^o_{i+1,N}(\omega)$; $\forall$ i, $\forall\ \omega\varepsilon[-\pi,\pi]$. Let $\underline{f}_s(\omega)$ be a spectral density matrix that belongs to class F_{L,ε_s}, and let $\{\lambda_{is}(\omega);\ 1 \leq i \leq n\}$ be the set of ordered eigenfunctions of the matrix $\underline{f}_s(\omega)$; where $\lambda_{is}(\omega) \geq \lambda_{i+1}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$; $\forall$ i. Then, the matrix $\underline{f}_s(\omega) - (1-\varepsilon_s)\underline{f}_{os}(\omega)$ is nonnegative definite for all ω in $[-\pi,\pi]$, which induces the necessary condition $\lambda_{is}(\omega) \geq (1-\varepsilon_s)\ \lambda^o_{is}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$, $\forall$ i. A necessary and sufficient condition that the matrix $\underline{f}_s(\omega)$ must satisfy, so that the matrix $\underline{f}_s(\omega) - (1-\varepsilon_s)\ \underline{f}_{os}(\omega)$ is nonnegative definite $\forall\ \omega\varepsilon[-\pi,\pi]$, will involve both the eigenfunctions and the eigenvectors of the matrix $\underline{f}_s(\omega)$. A sufficient condition for the satisfaction of the latter is given by the inequality $\lambda_{ns}(\omega) \geq (1-\varepsilon_s)\ \lambda^o_{is}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$. Similarly, if $\underline{f}_N(\omega)$ is a positive definite matrix, and $\{\lambda_{iN}(\omega)\ ;\ 1 \leq i \leq n\}$ is the set of its ordered eigen-

functions, it is sufficient that $\lambda_{nN}(\omega) \geq (1-\varepsilon_N)\ \lambda^o_{1N}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$, for the matrix $\underline{f}_N(\omega) - (1-\varepsilon_N)\ \underline{f}_{oN}(\omega)$ to be nonnegative definite for all ω in $[-\pi,\pi]$. Through the sufficient conditions $\lambda_{ns}(\omega) \geq (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$ and $\lambda_{nN}(\omega) \geq (1-\varepsilon_N)\ \lambda^o_{1N}(\omega)$; $\forall\ \omega\varepsilon[-\pi,\pi]$, we can now define two classes F'_{s,ε_s} and F'_{N,ε_N} that are contained respectively in F_{L,ε_s} and F_{L,ε_N}. Those two new classes are defined as follows.

$$F'_{s,\varepsilon_s} = \{\underline{f}_s(\omega) : \lambda_{ns}(\omega) \geq (1-\varepsilon_s)\ \lambda^o_{1s}(\omega);\ \forall\ \omega\varepsilon[-\pi,\pi] \text{ and}$$

$$\mathrm{tr} \int_{-\pi}^{\pi} \underline{f}_s(\omega)\ d\omega = \int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}(\omega)\ d\omega \leq W_1 \qquad \text{(3B)}$$

; where $\{\lambda^o_{is}(\omega)$; $1 \leq i \leq n\}$ the set of ordered eigenfunctions of the nominal spectral density matrix $\underline{f}_{os}(\omega)$ in F_{L,ε_s}, $\underline{f}_s(\omega)$ a positive definite matrix $\forall\ \omega\varepsilon[-\pi,\pi]$ and $\{\lambda_{is}(\omega)$; $1 \leq i \leq n\}$ the set of its ordered eigenfunctions$\}$.

$$F'_{N,\varepsilon_N} = \{\underline{f}_N(\omega) : \lambda_{ns}(\omega) \geq (1-\varepsilon_N)\ \lambda^o_{1N}(\omega);\ \forall\ \omega\varepsilon[-\pi,\pi] \text{ and}$$

$$\mathrm{tr} \int_{-\pi}^{\pi} \underline{f}_N(\omega)\ d\omega = \int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{iN}(\omega)\ d\omega \leq W_2 \qquad \text{(3C)}$$

; where $\{\lambda^o_{iN}(\omega)$; $1 \leq i \leq n\}$ the set of ordered eigenfunctions of the nominal spectral density matrix $\underline{f}_{oN}(\omega)$ in F_{L,ε_N}, $\underline{f}_N(\omega)$ a positive definite matrix $\forall\ \omega\varepsilon[-\pi,\pi]$, and $\{\lambda_{iN}(\omega)$; $1 \leq i \leq n\}$ the set of its ordered eigenfunctions$\}$.

The positive constants W_1 and W_2 in the classes F'_{s,ε_s} and F'_{N,ε_N} respectively, are the energy constraints in the optimization problem in (3A). The classes F'_{s,ε_s} and F'_{N,ε_N} are as class F in [8], and they are clearly convex and locally compact with respect to the metric in (2). The classes F'_{s,ε_s} and F'_{N,ε_N} are nonempty if respectively

$$\int_{-\pi}^{\pi} \lambda_{1s}^{o}(\omega)\; d\omega \le W_1\; n^{-1}(1-\varepsilon_s)^{-1} \text{ and} \int_{-\pi}^{\pi} \lambda_{1N}^{o}(\omega)\; d\omega \le W_2\; n^{-1}(1-\varepsilon_N)^{-1}.$$

Let us state the following optimization problem on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$.

Find the supremum of

$$e_b(\underline{f}_s,\underline{f}_N) \triangleq (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega) \right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega) \right]^{-1} \right)^{-1} d\omega$$

Subject to the constraints:

$$\lambda_{ns}(\omega) \ge (1-\varepsilon_s)\; \lambda_{1s}^{o}(\omega) \;\; ; \; \forall\; \omega\varepsilon[-\pi,\pi] \tag{3D}$$

$$\lambda_{nN}(\omega) \ge (1-\varepsilon_N)\; \lambda_{1N}^{o}(\omega) \;\; ; \; \forall\; \omega\varepsilon[-\pi,\pi]$$

$$\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}(\omega)\; d\omega \le W_1$$

$$\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{iN}(\omega)\; d\omega \le W_2$$

We can now express the main theorem for this section. Its proof is in the appendix.

<u>Theorem A</u>

Let the classes F_{L,ε_s} and F_{L,ε_N} of spectral density matrices be such that:

$$\underline{f}_{os}(\omega): \int_{-\pi}^{\pi} \lambda_{1s}^{o}(\omega)\; d\omega \le W_1 n^{-1}(1-\varepsilon_s)^{-1}$$

$$\underline{f}_{oN}(\omega): \int_{-\pi}^{\pi} \lambda_{1N}^{o}(\omega)\; d\omega \le W_2 n^{-1}(1-\varepsilon_N)^{-1} \tag{10}$$

Then, there exists a sufficient solution $\{\lambda^e_{is}(\omega)\ ;\ 1 \leq i \leq n\}$ and $\{\lambda^e_{iN}(\omega)\ ;\ 1 \leq i \leq n\}$ of the optimization problem (3D). This solution also provides a sufficient solution $(\underline{f}^e_s(\omega), \underline{f}^e_N(\omega))$ for the optimization problem (3A); where the eigenvectors of the matrices $\underline{f}^e_s(\omega)$ and $\underline{f}^e_N(\omega)$ are arbitrary, and it is such that:

1. If

$$J_{sN} \triangleq \int_{-\pi}^{\pi} \max(W_1^{-1} n(1-\varepsilon_s)\lambda^o_{1s}(\omega),\ W_2^{-1} n(1-\varepsilon_N)\lambda^o_{1N}(\omega))\ d\omega \leq 1 \tag{11}$$

Then,

$$\lambda^e_{is}(\omega) = \lambda^e_s(\omega)\ ;\ \forall\ i$$

$$\lambda^e_{iN}(\omega) = \lambda^e_N(\omega)\ ;\ \forall\ i \tag{A.1}$$

; where

$$\lambda^e_s(\omega) = W_2^{-1} W_1\ \lambda^e_N(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$$

$$\int_{-\pi}^{\pi} \lambda^e_s(\omega)\,d\omega = n^{-1}\ W_1$$

2. If

$$J_{sN} \triangleq \int_{-\pi}^{\pi} \max(W_1^{-1} n(1-\varepsilon_s)\lambda^o_{1s}(\omega),\ W_2^{-1} n(1-\varepsilon_N)\lambda^o_{1N}(\omega))\ d\omega > 1 \tag{12}$$

Then,

$$\lambda^e_{is}(\omega) = \lambda^e_s(\omega)\ ;\ \forall\ i$$

$$\lambda^e_{iN}(\omega) = \lambda^e_N(\omega)\ ;\ \forall\ i$$

; where

$$\lambda^e_s(\omega) = \begin{cases} \mu(1-\varepsilon_N)\ \lambda^o_{1N}(\omega)\ ;\ \omega\varepsilon E_\mu \triangleq \{\omega : \mu \geq (1-\varepsilon_s)(1-\varepsilon_N)^{-1} \dfrac{\lambda^o_{1s}(\omega)}{\lambda^o_{1N}(\omega)}\} \\[2ex] (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)\ ;\ \omega\varepsilon E^c_\mu \triangleq \{\omega : \mu < (1-\varepsilon_s)(1-\varepsilon_N)^{-1} \dfrac{\lambda^o_{1s}(\omega)}{\lambda^o_{1N}(\omega)}\} \end{cases} \tag{A.2}$$

$$\lambda_N^e(\omega) = \begin{cases} \nu(1-\varepsilon_s)\ \lambda_{1s}^o(\omega)\ ;\ \omega\varepsilon E_\nu \triangleq \{\omega : \nu \geq (1-\varepsilon_N)(1-\varepsilon_s)^{-1}\ \dfrac{\lambda_{1N}^o(\omega)}{\lambda_{1s}^o(\omega)}\} \\ (1-\varepsilon_N)\ \lambda_{1N}^o(\omega)\ ;\ \omega\varepsilon E_\nu^c \triangleq \{\omega : \nu < (1-\varepsilon_N)(1-\varepsilon_s)^{-1}\ \dfrac{\lambda_{1N}^o(\omega)}{\lambda_{1s}^o(\omega)}\} \end{cases}$$

$$\mu : \int_{E_\mu} \mu(1-\varepsilon_N)\ \lambda_{1N}^o(\omega)\ d\omega + \int_{E_\mu^c} (1-\varepsilon_s)\ \lambda_{1s}^o(\omega)\ d\omega = n^{-1}\ W_1$$

$$\nu : \int_{E_\nu} \nu(1-\varepsilon_s)\ \lambda_{1s}^o(\omega)\ d\omega + \int_{E_\nu^c} (1-\varepsilon_N)\ \lambda_{1N}^o(\omega)\ d\omega = n^{-1}\ W_2$$

and the constants μ and ν are both positive and unique.

The robust linear filter is then given by: $\underline{H}^e(\omega) = \lambda_s^e(\omega)\ [\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1}\ I;\omega\varepsilon[-\pi,\pi]$, where I the identity matrix.

We observe from theorem A, that if condition (11) holds, then any information process with identical spectral ordered eigenfunctions $\lambda_s^e(\omega)$, and any noise process with identical spectral ordered eigenfunctions $\lambda_N^e(\omega) = W_1^{-1}\ W_2\ \lambda_s^e(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$ provide a sufficient solution. If, on the other hand, condition (12) holds instead, then the sufficient solution is strictly determined by the maximum eigenfunctions $\lambda_{1s}^o(\omega)$ and $\lambda_{1N}^o(\omega)$ of the nominal information and noise processes. Since the conditions (11) and (12) are both determined by the interrelationship between the eigenfunctions $\lambda_{1s}^o(\omega)$ and $\lambda_{1N}^o(\omega)$, it is clear that in all cases the sufficient solution depends on the latter. Solution (A.2) is graphically exhibited in figure 1. We point that the solutions in theorem A are not unique. They are sufficient, however. That is, any other solution can not be superior, performancewise.

4. The Game Solution on $F_{L,\varepsilon} \times F_Q$

In the present section, we consider the case where $C_s = F_{L,\varepsilon_s}$; with energy constraint W_s, and $C_N = F_Q$. The case $C_s = F_Q$ and $C_N = F_{L,\varepsilon_N}$ is symmetric to the former; thus it will not be covered explicitly. As in section 3, and due to the fact that both the present classes C_s and C_N are convex and locally compact with respect to the metric in (2), the game reduces again to obtaining the supremum of the expression $e_m(\underline{f}_s, \underline{f}_N)$ in (6) on $F_{L,\varepsilon_s} \times F_Q$. Let us denote W_1 as in (3A), and let $\{\lambda_{is}(\omega)\}$, $\{\lambda_{iN}(\omega)\}$, be the ordered eigenfunctions for the information and noise processes, as in section 3. Let, as in section 3, $\{\lambda^o_{is}(\omega)\}$ be the set of ordered eigenfunctions of the nominal spectral density matrix $\underline{f}_{os}(\omega)$ in F_{L,ε_s}. The original optimization problem here consists of finding the supremum:

$$\sup_{\underline{f}_s \,\varepsilon\, F_{L,\varepsilon_s} \text{ and } \underline{f}_N \,\varepsilon\, F_Q} e_m(\underline{f}_s, \underline{f}_N) \tag{13}$$

; where the constraints induced by the class F_Q do not impose any restrictions on the eigenvectors.

Parallel to the optimization problem (3D) in section 3, let us state the following optimization problem on $F'_{s,\varepsilon_s} \times F_Q$ that involves ordered eigenfunctions only; where F'_{s,ε_s} as in section 3.

Find the supremum of

$$e_b(\underline{f}_s, \underline{f}_N) \triangleq (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega) \right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega) \right]^{-1} \right)^{-1} d\omega$$

subject to the constraints:

$$\lambda_{ns}(\omega) \geq (1-\varepsilon_s)\, \lambda^o_{1s}(\omega) \ ; \ \forall\, \omega \varepsilon [-\pi, \pi] \tag{4A}$$

$$\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}(\omega)\, d\omega \leq W_1$$

$$\int_{A_j} \sum_{i=1}^{n} \lambda_{iN}(\omega)\ d\omega = c_j\ ;\ j = 0,\dots,k$$

$$;\ \text{where } A_0 \triangleq [-\pi,\pi] - \bigcup_{j=1}^{k} A_j\ ,\quad c_0 \triangleq c - \sum_{j=1}^{k} c_j$$

As with theorem A in section 3, we can now express the following main theorem for this section. Its solution is in the appendix.

<u>Theorem B</u>

Let the class F_{L,ε_s} of spectral density matrices be such that:

$$\underline{f}_{os}(\omega)\ :\ \int_{-\pi}^{\pi} \lambda^{o}_{1s}(\omega)\ d\omega \leq W_1\ n^{-1}(1-\varepsilon_s)^{-1} \tag{14}$$

Then, there exists a sufficient solution $\{\lambda^e_{is}(\omega)\ ;\ 1 \leq i \leq n\}$ and $\{\lambda^e_{iN}(\omega)\ ;\ 1 \leq i \leq n\}$ of the optimization problem (4A). This solution also provides a sufficient solution $(\underline{f}^e_s(\omega),\ \underline{f}^e_N(\omega))$ for the optimization problem in (13); where the eigenvectors of the matrices $\underline{f}^e_s(\omega)$ and $\underline{f}^e_N(\omega)$ are arbitrary, and it is such that:

$$\lambda^e_{is}(\omega) = \lambda^e_s(\omega)\ ;\ \forall\ i$$

$$\lambda^e_{iN}(\omega) = \lambda^e_N(\omega)\ ;\ \forall\ i$$

$$;\ \text{where}\qquad \lambda^e_s(\omega) = \sum_{j=0}^{k} \frac{x_j}{x^o_j}\ (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)\cdot 1_{A_j}(\omega)$$

$$\lambda^e_N(\omega) = \sum_{j=0}^{k} \frac{c'_j}{x^o_j}\ (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)\cdot 1_{A_j}(\omega) \tag{B.1}$$

$$x^o_j = \int_{A_j} (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)\ d\omega\ ;\ j = 0,\dots,k$$

$$x_j = \begin{cases} \mu c'_j \ ; \text{ if } \mu c'_j \geq x^o_j \\ \int_{A_j} (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)\ d\omega \ ; \text{ if } \mu c'_j < x^o_j \end{cases}$$

$$\mu : \sum_{j:\mu c'_j \geq x^o_j} \mu c'_j + \sum_{j:\mu c'_j < x^o_j} x^o_j = n^{-1}\ W_1 \ , \ c'_j = n^{-1}\ c_j$$

and the constant μ is positive and unique.

The robust linear filter is then given by: $\underline{H}^e(\omega) = \lambda^e_s(\omega)\ [\ \lambda^e_s(\omega) + \lambda^e_N(\omega)]^{-1}\ I;\ \omega\varepsilon[-\pi,\pi]$, where I the identity matrix.

In theorem B above, $1_{A_j}(\omega)$ denotes the indicator function of the set A_j. From (B.1) we observe that the sufficient solution is such that the ratio $\dfrac{\lambda^e_s(\omega)}{\lambda^e_N(\omega)}$ is piecewise constant. This is exhibited graphically in figure 1. As in theorem A, the solution in thorem B is controlled by the maximum eigenfunction $\lambda^o_{1s}(\omega)$ of the nominal information process. The solution is also controlled here by the quantiles and the quantiled energy of the noise process.

5. The Game Solution on $F_Q \times F_Q$

In this section, we will consider the case where the convex and locally compact classes of information and noise processes are both F_Q, with different quantile and energy characteristics. Specifically, denoting $C_s = F_{Q_s}$ and $C_N = F_{Q_N}$, we define.

$$F_{Q_s} = \{\underline{f}_s(\omega) : \operatorname{tr} \int_{A_{si}} \underline{f}_s(\omega)\ d\omega = c_{si} \ ; \ i = 1,\ldots,k \tag{15}$$

$$; \text{ where } \{A_{si}\} \text{ measurable disjoint subsets of } [-\pi,\pi],\ \bigcup_{i=1}^{k} A_{si} = [-\pi,\pi],$$

$$\underline{f}_s(\omega) \text{ positive definite Hermitian matrix}\}$$

$$F_{Q_N} = \{\underline{f}_N(\omega) : \operatorname{tr} \int_{A_{Ni}} \underline{f}_N(\omega)\, d\omega = c_{Ni} \; ; \; i = 1,\ldots,m \tag{16}$$

$$; \text{ where } \{A_{Ni}\} \text{ measurable disjoint subsets of } [-\pi,\pi], \; \bigcup_{i=1}^{m} A_{Ni} = [-\pi,\pi],$$

$$\underline{f}_N(\omega) \text{ positive definite Hermitian matrix}\}$$

The game here reduces again to obtaining the supremum of expression $e_m(\underline{f}_s, \underline{f}_N)$ in (6) on $F_{Q_s} \times F_{Q_N}$. As in sections 3 and 4, we state the following optimization problem, that involves ordered eigenfunctions only.

Find the supremum of

$$e_b(\underline{f}_s, \underline{f}_N) \triangleq (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega) \right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega) \right]^{-1} \right)^{-1} d\omega$$

subject to the constraints:

$$\begin{aligned} &\int_{A_{sj}} \sum_{i=1}^{n} \lambda_{is}(\omega)\, d\omega = c_{sj} \; ; \; j = 1,\ldots,k \\ &\int_{A_{Nj}} \sum_{i=1}^{n} \lambda_{iN}(\omega)\, d\omega = c_{Nj} \; ; \; j = 1,\ldots,m \end{aligned} \tag{5A}$$

; where the sets $\{A_{Nj}\}$, $\{A_{sj}\}$ are given by the classes F_{Q_N} and F_{Q_s} respectively.

We now proceed with the main theorem in this section. Its proof is in the appendix.

<u>Theorem C</u>

Let $\mu(\cdot)$ denote the Lebesque measure in $[-\pi,\pi]$. Then, a sufficient solution $\{\lambda^e_{is}(\omega)\}$, $\{\lambda^e_{iN}(\omega)\}$ of the optimization problem (5A) is given by the following expressions.

$$\lambda_{is}^{e}(\omega) = \lambda_{s}^{e}(\omega) \; ; \; \forall \; i$$

$$\lambda_{iN}^{e}(\omega) = \lambda_{N}^{e}(\omega) \; ; \; \forall \; i \tag{C.1}$$

; where $\lambda_N^e(\omega) : \int_{A_{Nj}} \lambda_N^e(\omega)\, d\omega = n^{-1} c_{Nj}$; $j = 1,\ldots,m$, and arbitrary otherwise

$$\lambda_s^e(\omega) = n^{-1} \sum_{j=1}^{k} c_{sj} \left[\sum_{\ell=1}^{m} \int_{A_{sj} \cap A_{N\ell}} \lambda_N^e(\omega)\, d\omega \right]^{-1} \lambda_N^e(\omega) \cdot 1_{A_{sj}}(\omega)$$

A special solution evolves from (C.1), if:

$$\lambda_N^e(\omega) = n^{-1} \sum_{i=1}^{m} c_{Ni}\, \mu^{-1}(A_{Ni}) \cdot 1_{A_{Ni}}(\omega)$$

and then, (C.2)

$$\lambda_s^e(\omega) = n^{-1} \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{c_{sj}\, c_{Ni} \cdot 1_{A_{sj} \cap A_{Ni}}(\omega)}{\mu(A_{Ni}) \sum_{\ell=1}^{m} c_{N\ell}\, \mu(A_{N\ell} \cap A_{sj})\, \mu^{-1}(A_{N\ell})}$$

; where $1_A(\omega)$ denotes the indicator function of the set A.

A sufficient solution for the original optimization problem is given by the ordered eigenfunctions as in (C.1) or (C.2), and arbitrary eigenvectors. The robust linear filter is then given by:

$$\underline{H}^e(\omega) = \lambda_s^e(\omega)\, [\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1}\, I; \; \omega \varepsilon [-\pi,\pi], \text{ where I the identity matrix.}$$

From theorem C we observe that one of the eigenfunction sets (information or noise process) may be selected arbitrarily, but satisfying the corresponding quantiled energy constraints. Then, the remaining eigenfunction set is determined uniquely. A particular such selection is given by (C.2). Then, $\lambda_N^e(\omega)$ is a piecewise constant

function on the $\{A_{Ni}\}$ quantiles, and $\lambda_s^e(\omega)$ is a piecewise constant function on the $\{A_{si} \cap A_{Nj}\}$ quantiles. Solution (C.2) is exhibited graphically in figure 1. In the special case that $k = m$ and $A_{sj} = A_{Nj}$; $\forall$ j, the solution (C.1) gives $\lambda_s^e(\omega) = \sum_{j=1}^{k} c_{sj}\, c_{Nj}^{-1}\, 1_{A_{sj}}(\omega) \cdot \lambda_N^e(\omega)$, while the solution (C.2) corresponds to $\lambda_s^e(\omega)$ and $\lambda_N^e(\omega)$ that are both piecewise constant functions on the $\{A_{sj}\} = \{A_{Nj}\}$ quantiles, at different levels.

6. Performance Evaluation - Examples

In this section, we will discuss the performance induced by the robust linear filters in sections 3, 4, and 5. We will also present quantitative results for some examples.

The solutions in theorems A, B, and C provide the "worst" pair $(\underline{f}_s^e(\omega), \underline{f}_N^e(\omega))$ of information and noise spectral density matrices (in terms of performance) within the corresponding C_s and C_N classes, and they are all such that $\lambda_{is}^e(\omega) = \lambda_s^e(\omega)$; $\forall$ i, $\lambda_{iN}^e(\omega) = \lambda_N^e(\omega)$; $\forall$ i. Therefore, if I is the identity matrix, the spectral density matrices $\underline{f}_s^e(\omega)$ and $\underline{f}_N^e(\omega)$ are given by the following expressions.

$$\begin{aligned} \underline{f}_s^e(\omega) &= \lambda_s^e(\omega) \cdot I \ ; \ \omega \varepsilon [-\pi, \pi] \\ \underline{f}_N^e(\omega) &= \lambda_N^e(\omega) \cdot I \ ; \ \omega \varepsilon [-\pi, \pi] \end{aligned} \tag{17}$$

If $e(\underline{f}_s, \underline{f}_N, \underline{H})$ denotes the error induced by the spectral density matrices $\underline{f}_s$, $\underline{f}_N$, and the linear filter $\underline{H}$, and if $\underline{H}^e$ denotes the robust linear filter on $C_s \times C_N$, then in all three cases represented by theorems A, B, and C, we have:

$$\underline{H}^e(\omega) = \lambda_s^e(\omega) \ [\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1} \cdot I \ ; \ \omega \varepsilon [-\pi, \pi] \tag{18}$$

$$e(\underline{f}_s, \underline{f}_N, \underline{H}^e) \leq e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e) \ ; \ \forall (\underline{f}_s, \underline{f}_N) \ \varepsilon \ C_s \times C_N \tag{19}$$

$$e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e) = (2\pi)^{-1}\, n \int_{-\pi}^{\pi} \lambda_s^e(\omega) \cdot \lambda_N^e(\omega) \ [\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1} \, d\omega \tag{20}$$

; where I is the identity matrix, expression (20) is obtained by substituting expressions (17) in (6), and in (20) the dimensionality of the multivariable processes has been assumed equal to n. If we substitute expression (18) in expression (1), we obtain the mismatch error $e(\underline{f}_s, \underline{f}_N, \underline{H}^e)$ induced by the robust lienar filter $\underline{H}^e$ and some pair $(\underline{f}_s, \underline{f}_N)$ of spectral density matrices in $C_s \times C_N$. The mismatch error is then given by the following expression.

$$e(\underline{f}_s, \underline{f}_N, \underline{H}^e) = (2\pi)^{-1} \int_{-\pi}^{\pi} [\lambda_s^e(\omega)+\lambda_N^e(\omega)]^{-2}\{[\lambda_N^e(\omega)]^2\sum_i \lambda_{is}(\omega)+[\lambda_s^e(\omega)]^2 \sum_i \lambda_{iN}(\omega)\}d\omega$$

$$; \ (\underline{f}_s, \underline{f}_N) \in C_s \times C_N \qquad (21)$$

; where $\{\lambda_{is}(\omega)\}$ and $\{\lambda_{iN}(\omega)\}$ the ordered eigenfunctions of $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$ respectively.

Let $\underline{f}_{os}$ and $\underline{f}_{oN}$ be two nominal spectral density matrices in classes C_s and C_N respectively. Let $\underline{H}^e$ be the robust linear filter. Then, for all the classes in sections 3, 4, and 5, the mismatch error $e(\underline{f}_{os}, \underline{f}_{oN}, \underline{H}^e)$ is given by expression (21), if the spectral density matrices $\underline{f}_s$ and $\underline{f}_N$ are substituted respectively by $\underline{f}_{os}$ and $\underline{f}_{oN}$. Due to expression (6), the optimal error $e(\underline{f}_{os}, \underline{f}_{oN})$ at $(\underline{f}_{os}, \underline{f}_{oN})$ is given by the following expression.

$$e(\underline{f}_{os}, \underline{f}_{oN}) = (2\pi)^{-1} \operatorname{tr} \int_{-\pi}^{\pi} [\underline{f}_{os}^{-1}(\omega) + \underline{f}_{oN}^{-1}(\omega)]^{-1} d\omega \qquad (22)$$

Given the classes C_s and C_N, given nominal spectral density matrices $\underline{f}_{os}$ and $\underline{f}_{oN}$; in C_s and C_N respectively, given the robust linear filter $\underline{H}^e$ in $C_s \times C_N$, we will define two performance measures for the filter $\underline{H}^e$. The <u>efficiency</u> $E(\underline{H}^e, \underline{f}_s, \underline{f}_N, C_s, C_N)$ of the filter $\underline{H}^e$ at some pair $(\underline{f}_s, \underline{f}_N)$ in $C_s \times C_N$ is defined as the ratio of the optimal error $e(\underline{f}_s, \underline{f}_N)$ in (22), over the mismatch error $e(\underline{f}_s, \underline{f}_N, \underline{H}^e)$ in (21). That is,

$$E(\underline{H}^e, \underline{f}_s, \underline{f}_N, C_s, C_N) \triangleq e(\underline{f}_s, \underline{f}_N) \cdot e^{-1}(\underline{f}_s, \underline{f}_N, \underline{H}^e) \tag{23}$$

For given filter $\underline{H}^e$, the efficiency of $\underline{H}^e$ in $C_s \times C_N$, clearly attains its highest value, one, at $(\underline{f}_s^e, \underline{f}_N^e)$. If nominal spectral density matrices $\underline{f}_{os}$ and $\underline{f}_{oN}$ are given, an interesting measure is the efficiency of the filter $\underline{H}^e$ at $(\underline{f}_{os}, \underline{f}_{oN})$.

Given the robust filter $\underline{H}^e$, let us define,

$$e(C_s, C_N, \underline{H}^e) \triangleq \inf_{(\underline{f}_s, \underline{f}_N) \varepsilon C_s \times C_N} e(\underline{f}_s, \underline{f}_N, \underline{H}^e) \tag{24}$$

; where for any of the classes C_s, C_N in sections 3, 4, and 5, the error $e(\underline{f}_s, \underline{f}_N, \underline{H}^e)$ is given by (21).

Then, we define a second performance measure for the robust filter $\underline{H}^e$, in sections 3, 4, and 5. The <u>performance variation</u> $P(\underline{H}^e, C_s, C_N)$ of the filter $\underline{H}^e$ in $C_s \times C_N$ is defined as the ratio of the error $e(C_s, C_N, \underline{H}^e)$ in (24) over the error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$ in (20). That is,

$$P(\underline{H}^e, C_s, C_N) \triangleq e(C_s, C_N, \underline{H}^e) \cdot e^{-1}(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e) \tag{25}$$

The performance variation clearly does not exceed one (due to (19)), and it measures the maximum error variation induced by the robust linear filter $\underline{H}^e$ in $C_s \times C_N$.

The infimum represented by the error $e(C_s, C_N, \underline{H}^e)$ in (24) exists for compact classes C_s and C_N, and it can be computed explicitly for the cases studied in sections 3, 4, and 5. We state this result in a theorem, whose proof is in the appendix.

<u>Theorem 1</u>

The error $e(C_s, C_N, \underline{H}^e)$ in (24) is given respectively by the following expressions.

a. <u>$C_s \times C_N = F_{L,\varepsilon_s} \times F_{L,\varepsilon_N}$ in Section 3</u>

If condition (11) in theorem A is satisfied, then,

$$e(\mathcal{C}_s, \mathcal{C}_N, \underline{H}^e) = e(\underline{f}_s, \underline{f}_N, \underline{H}^e) = (2\pi)^{-1} W_1 W_2 (W_1 + W_2)^{-1}$$

$$; \forall(\underline{f}_s, \underline{f}_N) \in \mathcal{F}_{L,\varepsilon_s} \times \mathcal{F}_{L,\varepsilon_N} \qquad (26)$$

If condition (12) in theorem A is satisfied, then,

$$e(\mathcal{C}_s, \mathcal{C}_N, \underline{H}^e) = \varepsilon_s \frac{\nu^2}{(1+\nu)^2} W_s + \varepsilon_N \frac{\mu^2}{(1+\mu)^2} W_N +$$

$$+ (2\pi)^{-1} (1-\varepsilon_s) \int_{-\pi}^{\pi} \left[\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)}\right]^2 \sum_{i=1}^{n} \lambda_{is}^o(\omega)\, d\omega$$

$$+ (2\pi)^{-1} (1-\varepsilon_N) \int_{-\pi}^{\pi} \left[\frac{\lambda_s^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)}\right]^2 \sum_{i=1}^{n} \lambda_{iN}^o(\omega)\, d\omega \qquad (27)$$

; where W_s, W_N the energy constraints in classes $\mathcal{F}_{L,\varepsilon_s}$ and $\mathcal{F}_{L,\varepsilon_N}$ respectively, and μ, ν, $\lambda_N^e(\omega)$, $\lambda_s^e(\omega)$ as in solution (A.2) in theorem A.

b. $\underline{\mathcal{C}_s \times \mathcal{C}_N = \mathcal{F}_{L,\varepsilon_s} \times \mathcal{F}_Q}$ in Section 4

$$e(\mathcal{C}_s, \mathcal{C}_N, \underline{H}^e) = (2\pi)^{-1} (1-\varepsilon_s) \sum_{j=0}^{k} \left(\frac{c_j'}{c_j'+x_j}\right)^2 \int_{A_j} \sum_{i=1}^{n} \lambda_{is}^o(\omega)\, d\omega$$

$$+ (2\pi)^{-1} \sum_{j=0}^{k} \left(\frac{x_j}{c_j'+x_j}\right)^2 c_j'$$

$$+ \varepsilon_s \left[1 + n.\max_j (x_j\, c_j^{-1})\right]^{-1} W_s \qquad (28)$$

; where W_s the energy constraint in class $\mathcal{F}_{L,\varepsilon_s}$, $\{c_j'\}$ the energy levels in class $\mathcal{F}_Q$, times n^{-1}, and $\{x_j^o\}$, $\{x_j\}$, μ, as in theorem B.

c. $\underline{C_s \times C_N = F_Q \times F_Q}$ in Section 5

$$e(C_s, C_N, \underline{H}^e) = (2\pi)^{-1}\left[1 + \frac{c_{sj}}{\sum_{\ell=1}^{m} c_{N\ell}\,\mu(A_{N\ell}\cap A_{sj})\mu^{-1}(A_{N\ell})}\right]^{-2} c_{sj}$$

$$+ (2\pi)^{-1}\sum_{i=1}^{m} c_{Ni}\cdot \min_{j:A_{sj}\cap A_{Ni}\neq\emptyset}\left[1 + c_{sj}^{-1}\sum_{\ell=1}^{m} c_{N\ell}\,\mu(A_{N\ell}\cap A_{sj})\mu^{-1}(A_{N\ell})\right]^{-2}$$

(29)

; where all the quantities in expression (29) are as in theorem C.

In theorem 1 above, it has been assumed that in cases a. and b., the energy levels W_s and W_N are fixed.

From expressions (A.32), (A.38), and (A.42) in the appendix,the solutions (A.2), (B.1), and (C.2) in theorems A, B, and C respectively, and by direct substitution in expression (20), we also find,

1. <u>Case in theorem A. Condition (12) satisfied</u>

$$e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e) = (2\pi)^{-1} n \left\{(1-\varepsilon_N)\mu[1+\mu]^{-1}\int_{E_\mu} \lambda_{1N}^o(\omega)\, d\omega \; + \right.$$

$$\left. + (1-\varepsilon_s)\nu[1+\nu]^{-1}\int_{E_\nu} \lambda_{1s}^o(\omega)\,d\omega + \int_{E_\nu^c\cap E_\mu^c} \frac{(1-\varepsilon_s)(1-\varepsilon_N)\lambda_{1s}^o(\omega)\lambda_{1N}^o(\omega)}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)}\, d\omega\right\}$$

(30)

; where all the quantities in (30) are as in theorem A.

2. Case in theorem B.

$$e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e) = (2\pi)^{-1} n \sum_{j=0}^{k} c'_j \, x_j \, [c'_j + x_j]^{-1} \tag{31}$$

; where all the quantities in (31) are as in theorem B.

3. Case in theorem C. Solution (C.2)

$$e(\underline{f}_s^e,\underline{f}_N^e,\underline{H}^e)=(2\pi)^{-1}\sum_{i=1}^{m} c_{Ni}\mu^{-1}(A_{Ni}) \sum_{j=1}^{k}\left[1+c_{sj}^{-1}\sum_{\ell=1}^{m} c_{N\ell}\mu(A_{N\ell}\cap A_{sj})\mu^{-1}(A_{N\ell})\right]^{-1} \mu(A_{Ni}\cap A_{sj}) \tag{32}$$

; where all the quantities in (32) are as in theorem C.

Finally, for the robust filter $\underline{H}^e$ in sections 3 and 4, we can evaluate respectively a breakdown curve and a breakdown point. The breakdown curve for the robust filter $\underline{H}^e$ in section 3 is the set of points $(\varepsilon_s^*, \varepsilon_N^*)$ such that, if either one of the ε_s^*, ε_N^* values increases, then the error induced by $\underline{H}^e$ at any pair $(\underline{f}_s, \underline{f}_N)$ in $F_{L,\varepsilon_s} \times F_{L,\varepsilon_N}$ can take arbitrarily large values. The pairs $(\varepsilon_s, \varepsilon_N)$ for which that occurs are those that satisfy condition (11). Indeed, expression (26) is then satisfied, and the error $e(C_s, C_N, \underline{H}^e)$ increases then monotonically with the signal-to-noise ratio $W_1 W_2^{-1}$. The breakdown point for the robust filter $\underline{H}^e$ in section 4 is this value ε_s^* above which the error induced by $\underline{H}^e$ at any pair $(\underline{f}_s, \underline{f}_N)$ in $F_{L,\varepsilon_s} \times F_Q$ can be arbitrarily large. The values ε_s for which that occurs are such that $x_j = \mu c'_j$; $\forall j$ in the solution of theorem B. Indeed, by substitution in (21), we easily find that then the error $e(C_s, C_N, \underline{H}^e)$ is as in (26). The condition $x_j = \mu c'_j$; $\forall$ j provides, in general, a lower and an upper bound on ε_s. The upper bound ε_s^* is the breakdown point. The lower bound ε_{sL}, in conjuction with the lower bound ε'_{sL} provided by condition (14), determine acceptable ε_s regions, for which the solution in theorem B holds. Directly from (14) and (3A), we obtain:

$$\varepsilon'_{sL} = \left\{ 1 + 2\pi\, W_s \left[\int_{-\pi}^{\pi} n\, \lambda^o_{1s}(\omega)\, d\omega - 2\pi\, W_{os} \right]^{-1} \right\}^{-1} \tag{33}$$

From the condition $x_j = \mu\, c'_j$; $\forall$ j, and from the quantities in (B.1), we obtain after some simple computations,

$$\varepsilon_{sL} = \min_{m\varepsilon S_1} \left(\left[2\pi W_{os} - \left(\sum_{j=0}^{k} c'_j \right) \int_{A_m} n\, \lambda^o_{1s}(\omega)\, d\omega \right] \left[2\pi (W_{os} - W_s) - \left(\sum_{j=0}^{k} c'_j \right) \int_{A_m} n\, \lambda^o_{1s}(\omega)\, d\omega \right]^{-1} \right) \tag{34}$$

; where

$$S_1 \triangleq \left\{ i : \int_{A_i} n\, \lambda^o_{1s}(\omega)\, d\omega < 2\pi (W_{os} - W_s) \left[\sum_{j=0}^{k} c'_j \right]^{-1} \right\} \tag{35}$$

$$\varepsilon^*_s = \max_{m\varepsilon S_2} \left(\left[\left(\sum_{j=0}^{k} c'_j \right) \int_{A_m} n\lambda^o_{1s}(\omega)\, d\omega - 2\pi W_{os} \right] \left[\left(\sum_{j=0}^{k} c'_j \right) \int_{A_m} n\lambda^o_{1s}(\omega)\, d\omega - 2\pi (W_{os} - W_s) \right]^{-1} \right) \tag{36}$$

$$S_2 \triangleq \left\{ i : \int_{A_i} n\, \lambda^o_{1s}(\omega)\, d\omega > 2\pi\, W_{os} \left[\sum_{j=0}^{k} c'_j \right]^{-1} \right\} \tag{37}$$

The acceptable ε_s region for the case in theorem B is then $[\max(\varepsilon_{sL}, \varepsilon'_{sL}), \varepsilon^*_s]$. If class S_1 in (35) is empty, then $\varepsilon_{sL} = 0$. If class S_2 in (37) is empty, then $\varepsilon^*_s = 1$.

The breakdown curve induced by the robust linear filter $\underline{H}^e$ in section 3 is of particular interest, and it will be studied here analytically. As we will see, this curve behaves as the capacity region in information theoretic, two-user multiplexing.

Let, as in theorem A, n be the dimensionality of the vector information and noise processes, let $\lambda^{o}_{1s}(\omega)$ and $\lambda^{o}_{1N}(\omega)$ be the maximum nominal eigenfunctions for the information and noise processes respectively, let W_{os} and W_{oN} be the energies of the nominal information and noise processes, and let W_s and W_N signify the energies of the contaminating information and noise processes respectively. Let us define,

$$g(\omega) \triangleq \lambda^{o}_{1s}(\omega) \quad [\lambda^{o}_{1N}(\omega)]^{-1} \quad ; \; \omega \in [-\pi,\pi]$$

$$D \triangleq \left[\int_{-\pi}^{\pi} n \; \lambda^{o}_{1N}(\omega) \; d\omega\right]\left[\int_{-\pi}^{\pi} n \; \lambda^{o}_{1s}(\omega) \; d\omega\right]^{-1}$$

$$A_N \triangleq (2\pi)^2 \left[\int_{-\pi}^{\pi} n \; \lambda^{o}_{1s}(\omega) \; d\omega\right] \cdot \max \; (\min_{\omega} g(\omega), \; (2\pi)^{-2} \; D)$$

$$A_s \triangleq (2\pi)^2 \left[\int_{-\pi}^{\pi} n \; \lambda^{o}_{1N}(\omega) \; d\omega\right] \cdot \max \; (\min_{\omega} g^{-1}(\omega), \; (2\pi)^{-2} \; D^{-1}) \tag{38}$$

$$B_N \triangleq \left\{1 + 2\pi \; W_N \; [A_N - 2\pi \; W_{oN}]^{-1}\right\}^{-1}$$

$$B_s \triangleq \left\{1 + 2\pi \; W_s \; [A_s - 2\pi \; W_{os}]^{-1}\right\}^{-1}$$

We can now state the following theorem, whose proof is in the appendix.

<u>Theorem 2</u>

Let the function $g(\omega) = \lambda^{o}_{1s}(\omega) \; [\lambda^{o}_{1N}(\omega)]^{-1}$ be continuous everywhere in $[-\pi,\pi]$. Wherever differentiable, let the derivative of $g(\omega)$ be zero or infinity only on an ω set of measure zero. Then, the breakdown curve for the classes in theorem A is strictly concave, the acceptable $(\varepsilon_s, \varepsilon_N)$ region lies within the subplane $[B_s, 1]\text{x}[B_N, 1]$; where the numbers B_s, B_N are given by (38), and the derivatives $\left.\frac{\partial \varepsilon_s}{\partial \varepsilon_N}\right|_{\varepsilon_s = 1}$, $\left.\frac{\partial \varepsilon_N}{\partial \varepsilon_s}\right|_{\varepsilon_N = 1}$ are both zero.

The breakdown curve, and the acceptable region are exhibited in figure 2. If the function $g(\omega)$ in theorem 2 has zero and/or infinity derivatives on nonzero measure ω sets, then the breakdown curve will still be monotone, but it will not be strictly concave. We observe that the quantities $A_N - 2\pi W_{oN}$ and $A_s - 2\pi W_{os}$ in (38) are both nonnegative, and they are strictly controlled by the nominal information and noise processes. If both those quantities are strictly positive, the lower limits B_N and B_s in the acceptable region decrease monotonically as the contaminating energies W_N and W_s respectively increase. The limits B_N and B_s become zero for any W_N, W_s values, if respectively $A_N - 2\pi W_{oN} = 0$ and $A_s - 2\pi W_{os} = 0$. This, for example, occurs if $(2\pi)^{-2} \max_{\omega} g(\omega) \geq D \geq (2\pi)^2 \min_{\omega} g(\omega)$, and $\int_{-\pi}^{\pi} n\lambda_{1s}^{o}(\omega)d\omega = \int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}^{o}(\omega)d\omega$, $\int_{-\pi}^{\pi} n\lambda_{1N}^{o}(\omega)d\omega = \int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{iN}^{o}(\omega)d\omega$; where $g(\omega)$ and D are given by expressions (38). The above conditions reflect the case where the nominal ordered eigenfunctions (for both the information and noise processes) are equal to each other. We finally note that the lower boundaries B_s and B_N in the acceptable region of figure 2 are results of the conditions (10) in theorem A. Those conditions were imposed so that the robust solution is independent of spectral eigenvectors.

We will complete this section by presenting and quantitatively analyzing some examples. We will present examples for the two, more interesting, cases in sections 3, and 4. We will base our quantitative analysis on the performance measures introduced in this section.

a. <u>Example for Case in Section 3</u>

Let the nominal information and noise vector processes have dimensionality two. We select,

$$\underline{f}_{os}(\omega) = \begin{bmatrix} \lambda^o_{1s}(\omega) & 0 \\ 0 & \lambda^o_{2s}(\omega) \end{bmatrix}$$

$$\underline{f}_{oN}(\omega) = \begin{bmatrix} \lambda^o_{1N}(\omega) & 0 \\ 0 & \lambda^o_{2N}(\omega) \end{bmatrix} \tag{39}$$

; where

$$\lambda^o_{1s}(\omega) = \sigma^2_s[1 + 2\ \alpha_1 \cos\omega + \alpha^2_1]^{-1}$$

$$\lambda^o_{2s}(\omega) = \sigma^2_s[1 + 2\ \alpha_2 \cos\omega + \alpha^2_2]^{-1}$$

$$\lambda^o_{1N}(\omega) = \sigma^2_N[1 + 2\ \beta_1 \cos\omega + \beta^2_1]^{-1} \tag{40}$$

$$\lambda^o_{2N}(\omega) = \sigma^2_N[1 + 2\ \beta_2 \cos\omega + \beta^2_2]^{-1}$$

We also select $W_s\ W^{-1}_{os} = 1$ and $W_N\ W^{-1}_{oN} = 1$. Increased $W_s\ W^{-1}_{os}$ and $W_N\ W^{-1}_{oN}$ values will give uniformly inferior performance for any pair $(\varepsilon_s, \varepsilon_N)$. Decreased such values will give uniformly superior performance, instead. In expressions (40), we select,

$$\sigma^2_s = 5 \times 10^{-2}$$

$$\sigma^2_N = 10^{-2} \tag{41}$$

We also select,

$$\alpha_1 = \alpha_2 = .98$$

$$\beta_1 = \beta_2 = -.93 \tag{42}$$

The above example corresponds to the case of equal eigenfunctions for both the

noise and information nominal spectral density matrices. Thus, the acceptable region of figure 2 is here such that $B_s = B_N = 0$. Computed points of the breakdown curve are given by table 1 below.

ε_s	.05	.1	.16	.25	.34	.46	.58	.70	.79	.85	.91	.97
ε_N	.95	.88	.79	.70	.61	.49	.37	.25	.16	.10	.07	.01

Table 1

Breakdown Curve for Example (39)-(42)

For the example stated by expressions (39)-(42), we computed for various $(\varepsilon_s, \varepsilon_N)$ values, the efficiency $E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$ at the nominal pair $(\underline{f}_{os}, \underline{f}_{oN})$, the peformance variation $P(\underline{H}^e, C_s, C_N)$, and the error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$. We also computed the error variation induced by the optimal filter $\underline{H}^o$ at $(\underline{f}_{os}, \underline{f}_{oN})$, within the class $C_s \times C_N$. We denote this error variation $P(\underline{H}^o, C_s, C_N)$, and we define it as the ratio of $\min_{(\underline{f}_s, \underline{f}_N) \varepsilon C_s x C_N} e(\underline{f}_s, \underline{f}_N, \underline{H}^o)$ over $\max_{(\underline{f}_s, \underline{f}_N) \varepsilon C_s x C_N} e(\underline{f}_s, \underline{f}_N, \underline{H}^o)$. We exhibit our results in tables 2, 3, 4, and 5 respectively. In table 6, we exhibit values for the quantity

$$\max_{(\underline{f}_s, \underline{f}_N) \varepsilon C_s x C_N} e(\underline{f}_s, \underline{f}_N, \underline{H}^o) \triangleq e(\underline{H}^o).$$

ε_s \ ε_N	.002	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.85	.9	.95
.002	.986	.985	.981	.972	.959	.941	.893	.832	.761	.677	.578	.457	.381	.292	.187
.1	.174	.173	.169	.163	.158	.152	.141	.131	.120	.110	.101	.091	.087	.083	.083
.3	.109	.109	.107	.106	.104	.102	.098	.094	.091	.087	.083	.083	.083	.083	.083
.5	.097	.096	.096	.095	.093	.092	.089	.086	.083	.083	.083	.083	.083	.083	.083
.7	.091	.091	.091	.090	.088	.087	.083	.083	.083	.083	.083	.083	.083	.083	.083
.8	.09	.09	.089	.087	.085	.083	.083	.083	.083	.083	.083	.083	.083	.083	.083
.9	.089	.088	.087	.083	.083	.083	.083	.083	.083	.083	.083	.083	.083	.083	.083

Table 2

$E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$ for Example (39)-(42)

$\varepsilon_s \backslash \varepsilon_N$	.002	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.85	.9	.95
.002	.733	.680	.503	.378	.302	.252	.190	.156	.135	.124	.123	.139	.164	.219	.375
.1	.682	.681	.675	.671	.670	.672	.684	.706	.738	.782	.840	.914	.958	1.00	1.00
.3	.852	.852	.851	.852	.856	.860	.876	.897	.926	.962	1.00	1.00	1.00	1.00	1.00
.5	.903	.903	.903	.906	.912	.919	.940	.968	1.00	1.00	1.00	1.00	1.00	1.00	1.00
.7	.927	.928	.930	.937	.948	.963	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
.8	.935	.936	.941	.955	.977	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
.9	.942	.943	.960	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 3

$P(\underline{H}^e, C_s, C_N)$ for Example (39)-(42)

$\varepsilon_s \backslash \varepsilon_N$	.002	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.85	.9	.95
.002	.101	.109	.145	.189	.233	.277	.365	.449	.532	.613	.692	.768	.803	.837	.866
.1	.617	.621	.641	.666	.689	.712	.753	.789	.821	.846	.865	.877	.879	.880	.880
.3	.789	.791	.801	.813	.824	.834	.850	.864	.873	.879	.880	.880	.880	.880	.880
.5	.839	.840	.846	.854	.860	.866	.874	.879	.880	.880	.880	.880	.880	.880	.880
.7	.862	.863	.868	.873	.876	.879	.880	.880	.880	.880	.880	.880	.880	.880	.880
.8	.870	.871	.875	.878	.880	.880	.880	.880	.880	.880	.880	.880	.880	.880	.880
.9	.876	.877	.879	.880	.880	.880	.880	.880	.880	.880	.880	.880	.880	.880	.880

Table 4

$e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$ for Example (39)-(42)

ε_s \ ε_N	.002	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.85	.9	.95
.002	.797	.764	.633	.525	.451	.398	.325	.278	.245	.221	.202	.188	.181	.176	.170
.1	.077	.077	.077	.077	.076	.076	.075	.074	.074	.073	.072	.072	.072	.071	.071
.3	.028	.028	.028	.028	.029	.029	.030	.030	.031	.031	.032	.032	.032	.033	.033
.5	.017	.017	.017	.018	.018	.018	.019	.019	.020	.020	.020	.021	.021	.021	.022
.7	.013	.013	.013	.013	.013	.013	.014	.014	.015	.015	.015	.016	.016	.016	.016
.8	.011	.011	.011	.011	.012	.012	.012	.013	.013	.013	.014	.014	.014	.014	.014
.9	.010	.010	.010	.010	.010	.011	.011	.011	.012	.012	.012	.013	.013	.013	.013

Table 5

$P(\underline{H}^o, C_s, C_N)$ for Example (39)-(42)

ε_s \ ε_N	.002	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.85	.9	.95
.002	.141	.147	.181	.223	.265	.307	.390	.474	.558	.642	.726	.810	.852	.894	.935
.1	1.46	1.466	1.50	1.542	1.584	1.626	1.71	1.793	1.877	1.961	2.047	2.129	2.171	2.213	2.255
.3	4.152	4.159	4.192	4.234	4.276	4.318	4.402	4.486	4.569	4.653	4.737	4.821	4.863	4.905	4.947
.5	6.844	6.851	6.884	6.926	6.986	7.010	7.094	7.178	7.262	7.346	7.429	7.513	7.555	7.597	7.639
.7	9.536	9.543	9.577	9.618	9.660	9.702	9.786	9.870	9.954	10.038	10.122	10.205	10.247	10.289	10.331
.8	10.882	10.889	10.923	10.965	11.006	11.048	11.132	11.216	11.300	11.384	11.486	11.552	11.593	11.635	11.677
.9	12.228	12.235	12.269	12.311	12.353	12.395	12.478	12.562	12.646	12.730	12.814	12.898	12.940	12.981	13.023

Table 6

$e(\underline{H}^o)$ for Example (39)-(42)

We note that large $P(\underline{H}^e, C_s, C_N)$ and $P(\underline{H}^o, C_s, C_N)$ values correspond to small error deviations within $C_s \times C_N$, as induced by the filters $\underline{H}^e$ and $\underline{H}^o$ respectively. From tables 3 and 4, we observe that, for large values of the pair $(\varepsilon_s, \varepsilon_N)$, the error deviation within $C_s \times C_N$ is small, but it deviates around absolutely large values of the maximum error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$. From table 2 and 4, we observe that the efficiency at the nominal pair $(\underline{f}_{os}, \underline{f}_{oN})$, and the maximum error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$ both degradate gracefully as the values of the pair $(\varepsilon_s, \varepsilon_N)$ increase. The values .083, 1,00, and .880 in tables 2, 3, and 4 respectively correspond to values $(\varepsilon_s, \varepsilon_N)$ that are beyond the acceptable region; that is, beyond the breakdown curve. Comparing tables 3 and 4, with tables 5 and 6, we observe the truly dramatic effect of the robust filter $\underline{H}^e$. Indeed, from tables 5 and 6 we see clearly that if the presence of the contamination is ignored, and the optimal at the nominals filter is adopted, then the error fluctuation as well as the maximum error in $C_s \times C_N$ are dramatically large, as compared to those induced by the robust filter. This is so, even for small contamination parameters ε_s and ε_N. Thus, the results of the comparison between tables 3 and 4, and 5 and 6 speak for themselves, and those results are the true advocates in favor of the robust approach.

In figure 3, we have plotted the efficiency $E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$, the maximum error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$, and the performance variation $P(\underline{H}^e, C_s, C_N)$, for the example (39)-(42), as functions of ε_N, for $\varepsilon_s = .1$ and $\varepsilon_s = .7$. We notice that for both $\varepsilon_s = .1$ and $\varepsilon_s = .7$, the performance variation induced by the robust filter maintains uniformly high values, but so does the maximum induced error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$. Thus the error induced by $\underline{H}^e$ fluctuates relatively a little within $C_s \times C_N$, but it does so around relatively large error values. At $\varepsilon_s = .1$, the maximum error $e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$ is more than 25 percent less of the same error at $\varepsilon_s = .7$, for ε_N values not exceeding .01. The maximum gain in performance variation $P(\underline{H}^e, C_s, C_N)$, as one moves from $\varepsilon_s = .1$ to $\varepsilon_s = .7$ is about 33 percent, realized at about $\varepsilon_N = .18$. The gain in efficiency

$E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$, as one moves from $\varepsilon_s = .7$ to $\varepsilon_s = .1$, reduces monotonically as ε_N increases. The maximum such gain exceeds 8 percent. We note that the ε_N regions in figure 3, within which the curves (2), (3), and (4) converge to the respective values .083, 1.00, and .880, signify points beyond the breakdown curve. In figure 3, we have also plotted the performance variation $P(\underline{H}^o, C_s, C_N)$, for $\varepsilon_s = .1$. We observe how dramatically smaller than $P(\underline{H}^e, C_s, C_N)$ this variation is. This fact, in conjuction with the off-scale values of the maximum error $e(\underline{H}^o)$ exhibits nicely the superiority of the robust filter $\underline{H}^e$, as compared to the optimal at $(\underline{f}_{os}, \underline{f}_{oN})$ filter $\underline{H}^o$. At $\varepsilon_s = .7$, this superiority is far more striking, and it is deleted from figure 3.

We point out that if, in our example, we select the nominal eigenfunctions in (40) such that $\lambda^o_{1s}(\omega) \neq \lambda^o_{2s}(\omega)$ and $\lambda^o_{1N}(\omega) \neq \lambda^o_{2N}(\omega)$, the performance induced but the robust filter $\underline{H}^e$ will deteriorate uniformly. This deterioration will be increasing, as the distance between the corresponding eigenfunctions increases.

b. Example for Case in Section 4

Let the vector information and noise processes have dimensionality two. We select, as with the example for section 3,

$$\underline{f}_{os}(\omega) = \begin{bmatrix} \lambda^o_{1s}(\omega) & 0 \\ 0 & \lambda^o_{2s}(\omega) \end{bmatrix}$$

$$\begin{aligned} \lambda^o_{1s}(\omega) &= \lambda^o_{2s}(\omega) = \sigma_s^2[1 + 2\,\alpha \cos\omega + \alpha^2]^{-1} \\ \sigma_s^2 &= 5 \times 10^{-2} \\ \alpha &= .98 \\ W_s &= W_{os} \end{aligned} \qquad (43)$$

Then, ε'_{sL} in (33) is equal to zero, and the class S_1 in (35) is empty. Thus, $\max(\varepsilon'_{sL}, \varepsilon_{sL}) = 0$.

Let us select a nominal spectral density $\underline{f}_{oN}(\omega)$ in the class F_Q. We need this, to evaluate performance measures $P(\underline{H}^o, C_s, C_N)$ and $e(\underline{H}^o)$, as with the example for the case in section 3. Let $\underline{f}_{oN}(\omega)$ be such that,

$$\underline{f}_{oN}(\omega) = \begin{bmatrix} \lambda^o_{1N}(\omega) & 0 \\ 0 & \lambda^o_{2N}(\omega) \end{bmatrix}$$

$$\lambda^o_{1N}(\omega) = \lambda^o_{2N}(\omega) = \sigma^2_N[1 + 2\,\beta\cos\omega + \beta^2]^{-1} \qquad (44)$$

$$\sigma^2_N = 10^{-2}$$

$$\beta = -.93$$

We select the quantiles $\{A_j;\ 0 \leq j \leq k\}$ in F_Q such that, $A_j = [j\pi(k+1)^{-1}, (j+1)\pi(k+1)^{-1})$. The energy level c_j in A_j is then equal to $\int_{A_j} 2\lambda^o_{1N}(\omega)\, d\omega$, where $\lambda^o_{1N}(\omega)$ is given by (44). We select k+1 values equal to 2, 4, 10, 20, and 30. As in the example for the case in section 3, we compute for various ε_s values the quantities $E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$, $P(\underline{H}^e, C_s, C_N)$, $e(\underline{f}^e_s, \underline{f}^e_N, \underline{H}^e)$, $P(\underline{H}^o, C_s, C_N)$, and $e(\underline{H}^o)$. Our results are exhibited respectively in tables 7, 8, 9, 10, and 11; where for all cases the class S_2 in (37) is empty, thus $\varepsilon^*_s = 1$.

k+1 \ ε_s	.002	.01	.02	.1	.2	.3	.4	.5	.6	.7	.8	.9
2	.643	.568	.453	.173	.124	.109	.101	.096	.093	.091	.090	.088
4	.895	.733	.528	.173	.125	.109	.101	.097	.094	.092	.090	.089
10	.970	.759	.537	.174	.125	.109	.101	.097	.094	.092	.090	.089
20	.977	.766	.538	.174	.125	.109	.101	.097	.094	.092	.090	.089
30	.981	.778	.550	.178	.126	.110	.102	.097	.094	.092	.090	.089

Table 7

$E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$ for Example (43)-(44)

k+1 \ ε_s	.002	.01	.02	.1	.2	.3	.4	.5	.6	.7	.8	.9
2	.818	.565	.515	.686	.799	.853	.884	.904	.919	.929	.938	.946
4	.743	.480	.459	.685	.798	.852	.883	.903	.917	.928	.936	.943
10	.719	.468	.454	.683	.798	.852	.883	.903	.917	.927	.936	.942
20	.719	.464	.454	.638	.798	.852	.883	.903	.917	.927	.935	.942
30	.724	.467	.452	.676	.793	.848	.879	.900	.914	.925	.933	.940

Table 8

$P(\underline{H}^e, C_s, C_N)$ for Example (43)-(44)

k+1 \ ε_s	.002	.01	.02	.1	.2	.3	.4	.5	.6	.7	.8	.9
2	.139	.227	.313	.618	.737	.790	.821	.840	.854	.864	.871	.877
4	.110	.208	.301	.617	.736	.789	.819	.838	.852	.862	.870	.876
10	.105	.206	.299	.616	.735	.789	.819	.838	.852	.862	.870	.876
20	.104	.205	.299	.616	.735	.789	.819	.838	.852	.862	.870	.876
30	.103	.201	.294	.609	.730	.784	.815	.835	.849	.860	.868	.874

Table 9

$e(\underline{f}_s^e, \underline{f}_N^e, \underline{H}^e)$ for Example (43)-(44)

$k+1$ \ ε_s	.002	.01	.02	.1	.2	.3	.4	.5	.6	.7	.8	.9
2	.075	.072	.069	.049	.035	.026	.020	.016	.012	.010	.008	.006
4	.227	.208	.188	.104	.064	.044	.033	.025	.019	.015	.012	.010
10	.670	.543	.438	.167	.090	.059	.042	.032	.025	.020	.016	.013
20	.812	.639	.504	.181	.096	.063	.045	.034	.027	.021	.017	.014
30	.853	.672	.530	.191	.102	.067	.048	.036	.029	.023	.018	.015

Table 10

$P(\underline{H}^o, C_s, C_N)$ for Example (43)-(44)

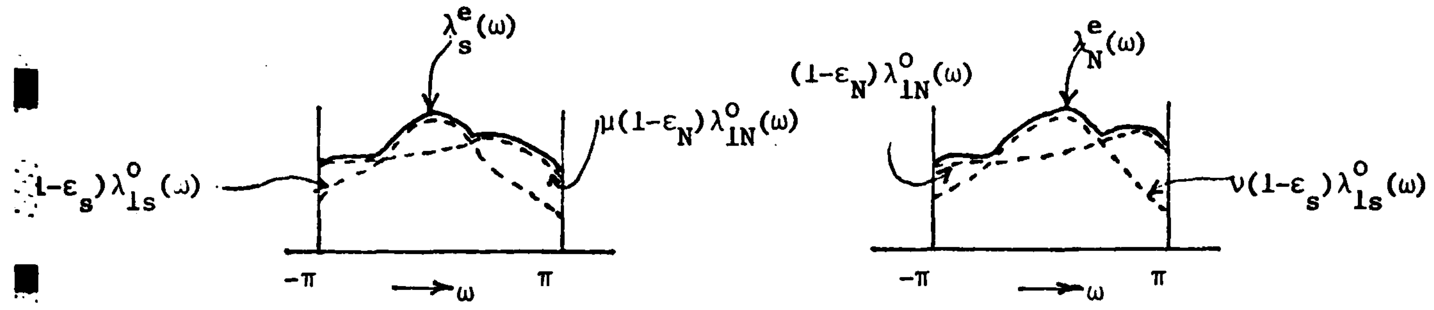
$k+1$ \ ε_s	.002	.01	.02	.1	.2	.3	.4	.5	.6	.7	.8	.9
2	.684	.706	.735	.960	1.242	1.525	1.807	2.089	2.371	2.653	2.935	3.217
4	.262	.285	.313	.538	.821	1.103	1.385	1.667	1.949	2.231	2.513	2.795
10	.099	.121	.149	.375	.657	.939	1.221	1.503	1.786	2.068	2.350	2.632
20	.085	.108	.136	.362	.644	.926	1.208	1.490	1.772	2.054	2.336	2.619
30	.082	.104	.131	.349	.621	.893	1.165	1.437	1.710	1.982	2.254	2.526

Table 11

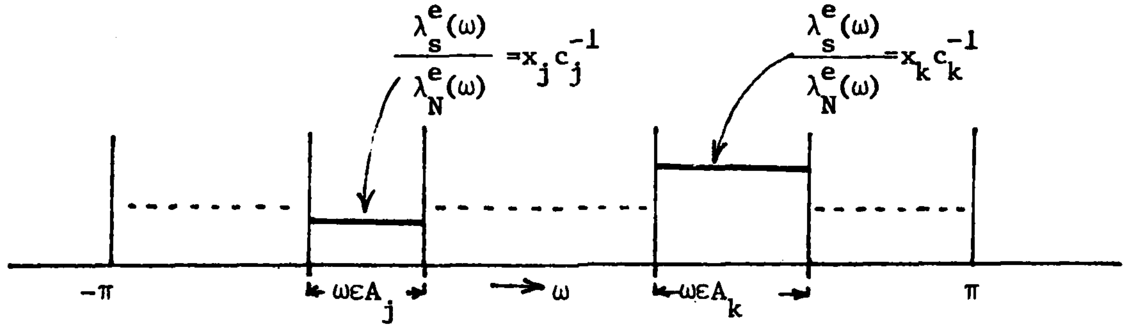
$e(\underline{H}^o)$ for Example (43)-(44)

From tables 7, 8, and 9, we observe that the number of the quantiles affects the performance induced by the robust filter $\underline{H}^e$, only for ε_s values below .02. Then, the performance improves as the number of the quantiles increases, with highest effect on the efficiency $E(\underline{H}^e, \underline{f}_{os}, \underline{f}_{oN}, C_s, C_N)$ and for $\varepsilon_s < .01$. The reason for this behavior is that, the more accurate representation of the noise spectral density matrix $\underline{f}_{oN}$ resulting from the increased number of quantiles, drives the robust filter $\underline{H}^e$ close to $\underline{H}^o$, only for small contamination ε_s. For relatively large such contamination, the

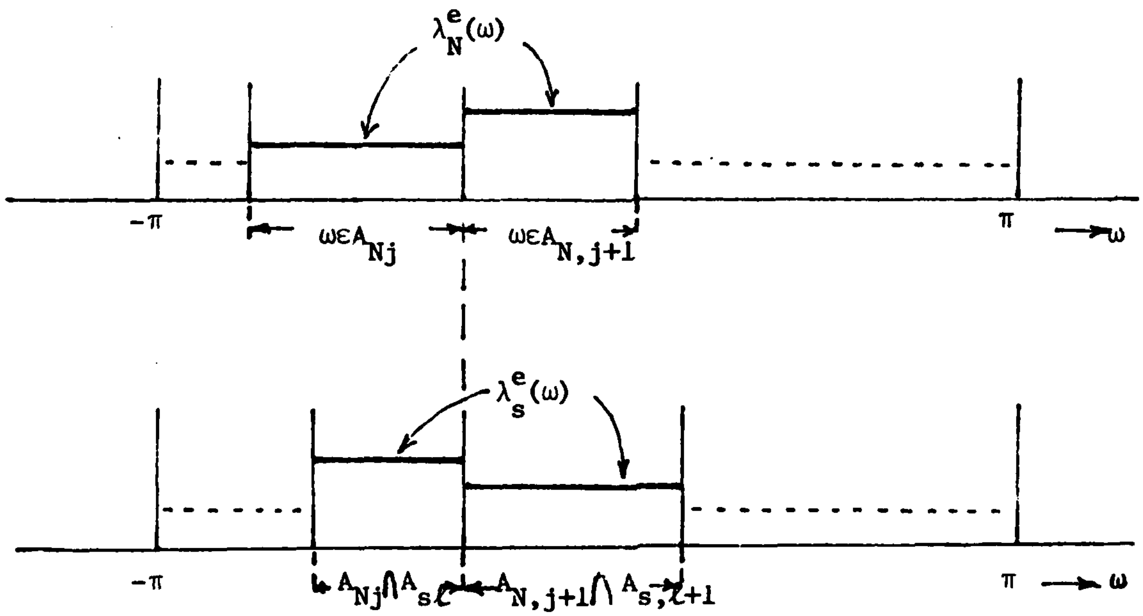
accuracy in the representation of $\underline{f}_{oN}$ has little effect on the design of the robust filter $\underline{H}^e$. Comparing tables 8 and 9, with tables 10 and 11, we see again the striking effect of the robust filter, as compared to the optimal at the nominals $(\underline{f}_{os}, \underline{f}_{oN})$ filter, $\underline{H}^o$.



Solution (A.2) - Theorem A

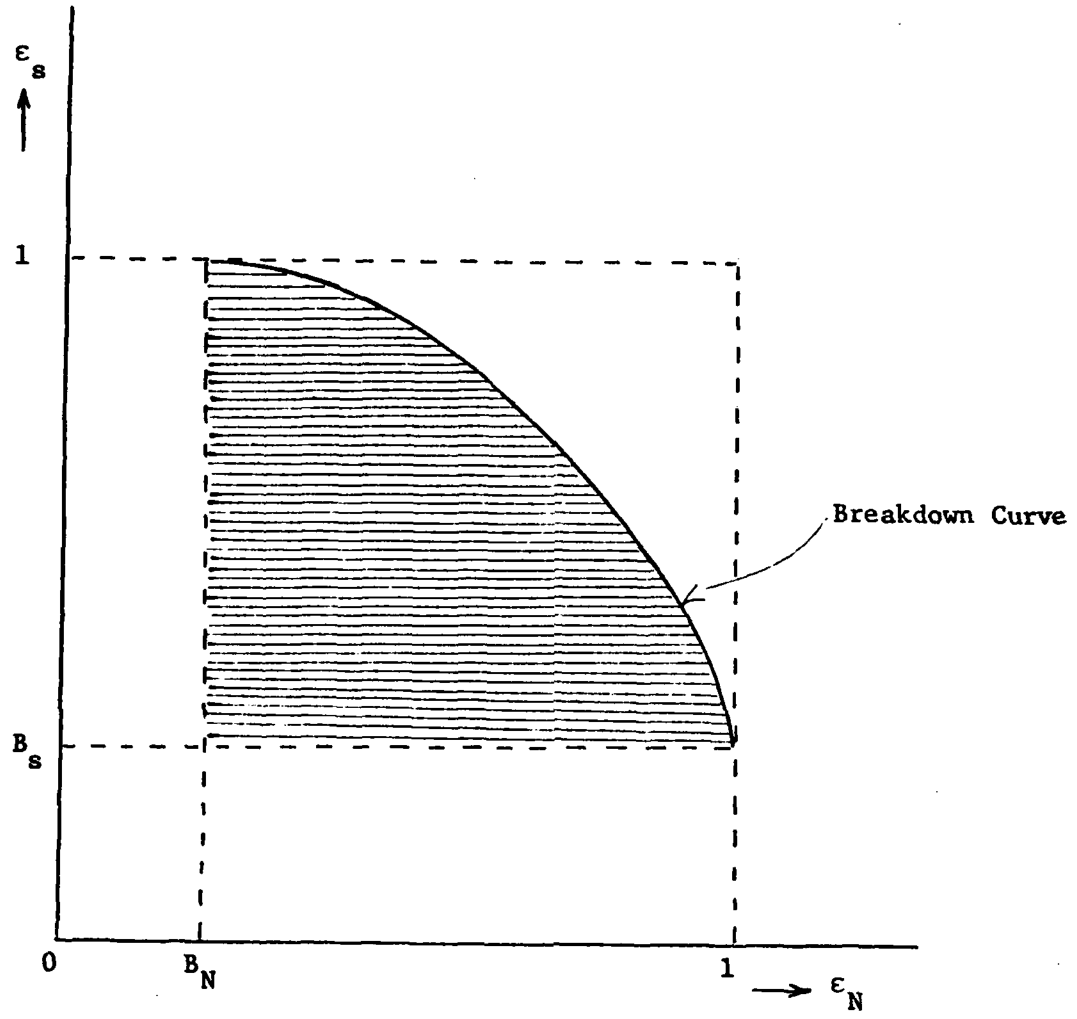


Solution (B.1) - Theorem B



Solution (C.1) - Theorem C

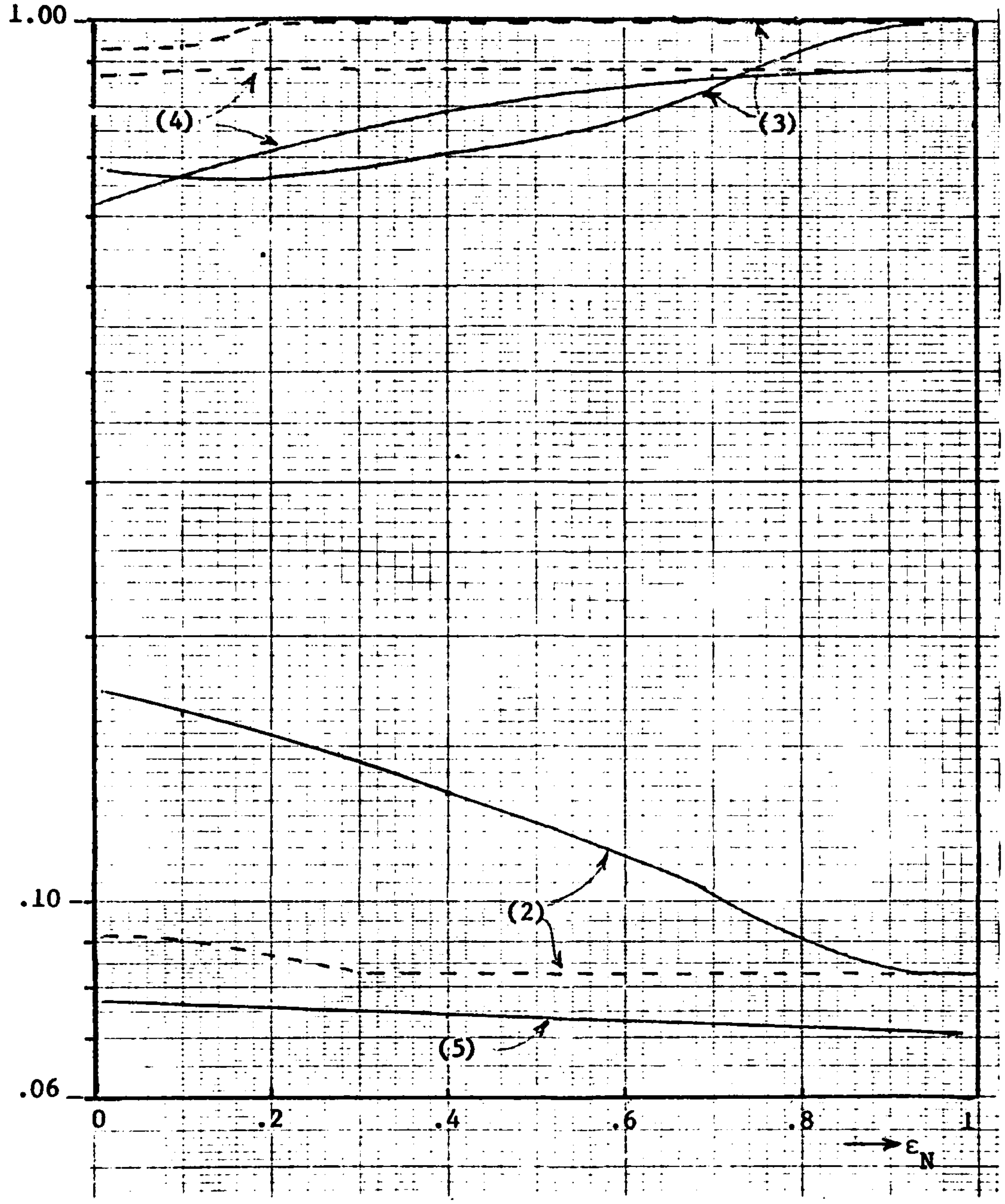
Figure 1



Shaded Area: Acceptable Region

Figure 2

Case in Theorem A: Breakdown Curve



(2): Efficiency from Table 2.
(3): Performance Variation from Table 3.
(4): Maximum Error from Table 4.
(5): Performance Variation from Table 5.

Solid Lines: $\varepsilon_s = .1$

Dashed Lines: $\varepsilon_s = .7$

Figure 3

Performance Curves. Example (39)-(42)

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Appendix

Proof of the Lemma

Define the set $A_g \triangleq \{x : x \in A,\ g(x) > 0\}$. If $f + g > 0$, a.e. in A, then $f > 0$, a.e. in $A - A_g$. Define,

$$\mu_g(B) \triangleq \int_B g\, dm \ ;\ B \subset A_g$$

$$\mu_f(B) \triangleq \int_B f\, dm \ ;\ B \subset [A - A_g] \tag{A.1}$$

$$\phi\,(\lambda) \triangleq \lambda[1+\lambda]^{-1} \ ;\ \lambda \in [-1,\infty)$$

The measures μ_g and μ_f in (A.1) are positive measures, and the function ϕ is clearly concave. We now write,

$$\int_A \frac{fg}{f+g}\, dm = \int_{A_g} g \cdot \phi\left(\frac{f}{g}\right) dm + \int_{A-A_g} f \cdot \phi\left(\frac{g}{f}\right) dm =$$

$$= \int_{A_g} \phi\left(\frac{f}{g}\right) d\,\mu_g + \int_{A-A_g} \phi\left(\frac{g}{f}\right) d\,\mu_f \tag{A.2}$$

Since the function ϕ is concave, we apply Jensen's inequality in (A.2), and we obtain:

$$\int_A \frac{fg}{f+g}\, dm \le \mu_g(A_g) \cdot \phi\left(\mu_g^{-1}(A_g) \int_{A_g} \frac{f}{g}\, d\,\mu_g\right)$$

$$+ \mu_f(A-A_g) \cdot \phi\left(\mu_g^{-1}(A-A_g) \int_{A-A_g} \frac{g}{f}\, d\,\mu_f\right) =$$

$$= \frac{\mu_g(A_g)\int_{A_g} \frac{f}{g}\, d\,\mu_g}{\mu_g(A_g) + \int_{A_g} \frac{f}{g}\, d\,\mu_g} + \frac{\mu_f(A-A_g)\int_{A-A_g} \frac{g}{f}\, d\,\mu_f}{\mu_f(A-A_g) + \int_{A-A_g} \frac{g}{f}\, d\,\mu_f} =$$

$$= \frac{\int_{A_g} gdm \cdot \int_{A_g} fdm}{\int_{A_g} gdm + \int_{A_g} fdm} + \frac{\int_{A-A_g} gdm \cdot \int_{A-A_g} fdm}{\int_{A-A_g} gdm + \int_{A-A_g} fdm} \leq \frac{\int_A gdm \cdot \int_A fdm}{\int_A fdm + \int_A gdm} \tag{A.3}$$

with equality in all parts iff $f = cg$; a.e. in A

The proof of the lemma is now complete.

<u>Proof of Corollary 2</u>

Let $\lambda_i(A)$, $\lambda_i(B)$, $\lambda_i(A^{-1} + B^{-1})$, $i = 1,\dots,n$ be the ordered eigenvalues of A, B and $A^{-1} + B^{-1}$ respectively. Let $\{\underline{a}_i, i=1,\dots n\}$, $\{\underline{b}_i, i=1,\dots n\}$, $\{\underline{c}_i, i=1,\dots n\}$ be the associated orthonormal sets of eigenvectors of A, B and $A^{-1} + B^{-1}$. Then, we can write:

$$A^{-1} = \sum_{i=1}^{n} \lambda_i^{-1}(A)\ \underline{a}_i\ \underline{a}_i^{*T}$$

$$B^{-1} = \sum_{i=1}^{n} \lambda_i^{-1}(B)\ \underline{b}_i\ \underline{b}_i^{*T}$$

and

$$\operatorname{tr}(A^{-1} + B^{-1})^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i(A^{-1}+B^{-1})} = \sum_{i=1}^{n} \frac{1}{\underline{c}_i^{*T}(A^{-1}+B^{-1})\underline{c}_i} =$$

$$= \sum_{i=1}^{n} \frac{1}{\sum_{k=1}^{n} \lambda_k^{-1}(A)(\underline{c}_i^{*T}\ \underline{a}_k)^2 + \sum_{\ell=1}^{n} \lambda_\ell^{-1}(B)(\underline{c}_i^{*T}\ \underline{b}_\ell)^2} \tag{A.4}$$

The relationships

$$\sum_{k=1}^{n} (\underline{c}_i^{*T}\ \underline{a}_k)^2 = \sum_{i=1}^{n} (\underline{c}_i^{*T}\ \underline{a}_k)^2 = 1\ ;\quad \forall\ k,\ \forall\ i$$

$$\sum_{\ell=1}^{n} (\underline{c}_i^{*T}\ \underline{b}_\ell)^2 = \sum_{i=1}^{n} (\underline{c}_i^{*T}\ \underline{b}_k)^2 = 1\ ;\ \forall\ k,\ \forall\ i$$

and the convexity of the last expression in (A.4) with respect to $\lambda_k^{-1}(A)$, $\lambda_\ell^{-1}(B)$ yield:

$$\mathrm{tr}(A^{-1} + B^{-1})^{-1} \leq \sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{\sum_{i=1}^{n} (\underline{c}_i^{*T}\ \underline{a}_k)^2\ (\underline{c}_i^{*T}\ \underline{b}_\ell)^2}{\lambda_k^{-1}(A) + \lambda_\ell^{-1}(B)} \tag{A.5}$$

If we apply the lemma for

$$f = \sum_{k=1}^{n}\ \sum_{\ell=1}^{n} \lambda_k(A)\ 1_{A_{k\ell}}$$

$$g = \sum_{k=1}^{n}\ \sum_{\ell=1}^{n} \lambda_\ell(B)\ 1_{A_{k\ell}}$$

and positive measure $m(\cdot)$ such that

$$m(A_{k\ell}) = \sum_{i=1}^{n} (\underline{c}_i^{*T}\ \underline{a}_k)^2\ (\underline{c}_i^{*T}\ \underline{b}_\ell)^2$$

; where $\{A_{k\ell}\}$ any sets: $\bigcup_{k=1}^{n}\ \bigcup_{\ell=1}^{n} A_{k\ell} = A$, $A_{k\ell}$ mutually exclusive, we obtain:

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n}\frac{\sum_{i=1}^{n}(\underline{c}_i^{*T}\,\underline{a}_k)^2\,(\underline{c}_i^{*T}\,\underline{b}_\ell)^2}{\lambda_k^{-1}(A)+\lambda_\ell^{-1}(B)}\le$$

$$\le\left[\left(\sum_{k=1}^{n}\sum_{\ell=1}^{n}\lambda_k(A)\sum_{i=1}^{n}(\underline{c}_i^{*T}\,\underline{a}_k)^2\,(\underline{c}_i^{*T}\,\underline{b}_\ell)^2\right)^{-1}+\right.$$

$$\left.+\left(\sum_{k=1}^{n}\sum_{\ell=1}^{n}\lambda_\ell(B)\sum_{i=1}^{n}(\underline{c}_i^{*T}\,\underline{a}_k)^2\,(\underline{c}_i^{*T}\,\underline{b}_\ell)^2\right)^{-1}\right]^{-1}=$$

$$=\left[\left(\sum_{k=1}^{n}\lambda_k(A)\right)^{-1}+\left(\sum_{\ell=1}^{n}\lambda_\ell(B)\right)^{-1}\right]^{-1}=$$

$$=\left[(\mathrm{tr}A)^{-1}+(\mathrm{tr}B)^{-1}\right]^{-1}\qquad\text{(A.6)}$$

From (A.4), (A.5), and (A.6) we conclude the desired result.

Proof of Theorem A

We will present the proof in two parts. In part 1, we will prove that (A.1) and (A.2) are sufficient solutions for the optimization problem (3D), and that they satisfy the supremum of $e_m(\underline{f}_s, \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$. In part 2, we will prove that a solution on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$ is sufficient for the optimization problem on $F_{L,\varepsilon_s} \times F_{L,\varepsilon_N}$.

Part 1

Let us define:

$$\lambda_s(\omega)\triangleq n^{-1}\sum_{i=1}^{n}\lambda_{is}(\omega)$$

$$\lambda_N(\omega)\triangleq n^{-1}\sum_{i=1}^{n}\lambda_{iN}(\omega)\qquad\text{(A.7)}$$

1. If condition (11) holds, then,

$$n^{-1} W_1 = \int_{-\pi}^{\pi} \lambda_s(\omega)\ d\omega \geq \int_{-\pi}^{\pi} \max\left((1-\varepsilon_s)\ \lambda_{1s}^{o}(\omega),\ W_2^{-1} W_1 (1-\varepsilon_N)\ \lambda_{1N}^{o}(\omega)\right)\ d\omega$$

$$n^{-1} W_2 = \int_{-\pi}^{\pi} \lambda_N(\omega)\ d\omega \geq \int_{-\pi}^{\pi} \max\left(W_1^{-1} W_2 (1-\varepsilon_s)\ \lambda_{1s}^{o}(\omega),\ (1-\varepsilon_N)\ \lambda_{1N}^{o}(\omega)\right) d\omega \tag{A.8}$$

The relationships in (A.8) show that there exist admissible eigenfunctions $\lambda_s(\omega)$ and $\lambda_N(\omega)$, as in (A.7), such that:

$$\lambda_s(\omega) \geq (1-\varepsilon_s)\ \lambda_{1s}^{o}(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$$

$$W_2^{-1} W_1\ \lambda_N(\omega) > (1-\varepsilon_s)\ \lambda_{1s}^{o}(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$$

$$\lambda_N(\omega) \geq (1-\varepsilon_N)\ \lambda_{1N}^{o}(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$$

$$W_1^{-1} W_2\ \lambda_s(\omega) \geq (1-\varepsilon_N)\ \lambda_{1N}^{o}(\omega)\ ;\ \forall\ \omega\varepsilon[-\pi,\pi]$$

Therefore, a choice of eigenfunctions as in (A.1) is fully consistent with the constraints of the optimization problem (3D). Also, the value of the objective function $e_b(\underline{f}_s, \underline{f}_N)$ in the problem (3D) becomes then,

$$e_b(\underline{f}_s^e, \underline{f}_N^e) = (2\pi)^{-1} W_1 W_2 (W_1 + W_2)^{-1} \tag{A.9}$$

Inversely, applying the lemma and corollary 1, we obtain:

$$e_b(\underline{f}_s, \underline{f}_N)$$

$$= (2\pi)^{-1} \int_{-\pi}^{\pi} \left[\sum_{i=1}^{n} \lambda_{is}(\omega)\right]\left[\sum_{i=1}^{n} \lambda_{iN}(\omega)\right]\left[\sum_{i=1}^{n} \lambda_{is}(\omega) + \sum_{i=1}^{n} \lambda_{iN}(\omega)\right]^{-1} d\omega \leq$$

$$\leq (2\pi)^{-1} \left[\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}(\omega)\, d\omega\right]\left[\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{iN}(\omega)\, d\omega\right]\left[\int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{is}(\omega)\, d\omega + \int_{-\pi}^{\pi} \sum_{i=1}^{n} \lambda_{iN}(\omega)\, d\omega\right]^{-1} =$$

$$= (2\pi)^{-1} W_1 W_2 (W_1 + W_2)^{-1} \tag{A.10}$$

From (A.9) and (A.10) we clearly conclude the sufficiency of the solution (A.1) in theorem A, for the optimization problem (3D).

2. Let condition (12) hold. Considering the functions in (A.7), and the objective function $e_b(\underline{f}_s, \underline{f}_N)$ in the optimization problem (3D), we have due to the lemma:

$$(2\pi)\ e_b(\underline{f}_s, \underline{f}_N) \leq n \int_{-\pi}^{\pi} \lambda_s(\omega)\cdot\lambda_N(\omega)\ [\lambda_s(\omega) + \lambda_N(\omega)]^{-1}\ d\omega \qquad (A.11)$$

On the other hand, for the admissible eigenfunctions $\{\lambda^e_{is}(\omega)\}$ and $\{\lambda^e_{iN}(\omega)\}$ in solution (A.2), we have:

$$(2\pi)\ e_b(\underline{f}^e_s, \underline{f}^e_N) = n \int_{-\pi}^{\pi} \lambda^e_s(\omega)\cdot\lambda^e_N(\omega)\ [\lambda^e_s(\omega) + \lambda^e_N(\omega)]^{-1}\ d\omega \qquad (A.12)$$

We will prove that for any admissible functions $\lambda_s(\omega)$, $\lambda_N(\omega)$, as in (A.7), we have:

$$\int_{-\pi}^{\pi} \lambda_s(\omega)\ \lambda_N(\omega)\,[\lambda_s(\omega) + \lambda_N(\omega)]^{-1}\ d\omega \leq \int_{-\pi}^{\pi} \lambda^e_s(\omega)\ \lambda^e_N(\omega)\,[\lambda^e_s(\omega) + \lambda^e_N(\omega)]^{-1}\ d\omega \quad (A.13)$$

We first observe that due to the condition in (12), the sufficient solution in (A.2) is such that:

$$\mu < W_2^{-1}\ W_1,\ \nu < W_1^{-1}\ W_2 \rightarrow \mu\ \nu < 1 \qquad (A.14)$$

Due to (A.14) we also conclude that the sets E_μ, E_ν in (A.2) are such that:

$$E_\mu \cap E_\nu = \emptyset \qquad (A.15)$$

$$E_\mu \subset E^c_\nu,\ E_\nu \subset E^c_\mu \qquad (A.16)$$

(A.15) is true, because if not, there would exist some ω in $E_\mu \cap E_\nu$ such that $\mu \geq (1-\varepsilon_s)(1-\varepsilon_N)^{-1} \dfrac{\lambda^o_{is}(\omega)}{\lambda^o_{1N}(\omega)}$ and $\nu \geq (1-\varepsilon_N)(1-\varepsilon_s)^{-1} \dfrac{\lambda^o_{1N}(\omega)}{\lambda^o_{1s}(\omega)}$, which contradicts (A.14).

(A.16) follows from (A.15). Due to (A.15) and (A.16), we conclude that the interval $[-\pi,\pi]$ can be subdivided into the three disjoint and exhaustive sets E_μ, E_ν, $E_\mu^c \cap E_\nu^c$. Finally, from the solution in (A.2), and the constraints in (3D), we clearly obtain that for any admissible functions $\lambda_s(\omega)$ and $\lambda_N(\omega)$, as in (A.7), we have:

$$\begin{aligned} \lambda_s(\omega) - \lambda_s^e(\omega) &\geq 0 \ ; \ \forall\, \omega\varepsilon\, E_\mu^c \\ \lambda_N(\omega) - \lambda_N^e(\omega) &\geq 0 \ ; \ \forall\, \omega\varepsilon\, E_\nu^c \end{aligned} \qquad \text{(A.17)}$$

We now concentrate on the proof of (A.13). Due to (A.14) - (A.16), we easily obtain:

$$\int_{-\pi}^{\pi} \lambda_s(\omega)\ \lambda_N(\omega)\,[\lambda_s(\omega) + \lambda_N(\omega)]^{-1}\, d\omega - \int_{-\pi}^{\pi} \lambda_s^e(\omega)\ \lambda_N^e(\omega)\,[\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1}\, d\omega =$$

$$= \int_{E_\mu} \frac{[\lambda_s(\omega)-\lambda_s^e(\omega)]\,[\lambda_N(\omega)-\lambda_N^e(\omega)] + \mu^2(1+\mu)^{-1}(1-\varepsilon_N)\lambda_{1N}^o(\omega)\,[\lambda_N(\omega)-\lambda_N^e(\omega)] + (\mu+1)^{-1}(1-\varepsilon_N)\lambda_{1N}^o(\omega)\,[\lambda_s(\omega)-\lambda_s^e(\omega)]}{(\mu+1)(1-\varepsilon_N)\ \lambda_{1N}^o(\omega) + [\lambda_s(\omega) - \lambda_s^e(\omega)] + [\lambda_N(\omega) - \lambda_N^e(\omega)]}\, d\omega$$

$$+ \int_{E_\mu^c \cap E_\nu^c} \frac{[\lambda_s(\omega)-\lambda_s^e(\omega)]\,[\lambda_N(\omega)- \lambda_N^e(\omega)] + \dfrac{[(1-\varepsilon_s)\lambda_{1s}^o(\omega)]^2\,[\lambda_N(\omega)-\lambda_N^e(\omega)]}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)} + \dfrac{[1-\varepsilon_N)\lambda_{1N}^o(\omega)]^2\,[\lambda_s(\omega)-\lambda_s^e(\omega)]}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)}}{(1-\varepsilon_s)\lambda_{1s}^o(\omega) + (1-\varepsilon_N)\ \lambda_{1N}^o(\omega) + [\lambda_s(\omega) - \lambda_s^e(\omega)] + [\lambda_N(\omega) - \lambda_N^e(\omega)]}\, d\omega$$

$$+ \int_{E_\nu} \frac{[\lambda_s(\omega)-\lambda_s^e(\omega)]\,[\lambda_N(\omega)-\lambda_N^e(\omega)] + (\nu+1)^{-1}(1-\varepsilon_s)\lambda_{1s}^o(\omega)\,[\lambda_N(\omega)-\lambda_N^e(\omega)] + \nu^2(\nu+1)^{-1}(1-\varepsilon_s)\lambda_{1s}^o(\omega)\,[\lambda_s(\omega)-\lambda_s^e(\omega)]}{(\nu+1)(1-\varepsilon_N)\lambda_{1s}^o(\omega) + [\lambda_s(\omega)-\lambda_s^e(\omega)] + [\lambda_N(\omega)-\lambda_N^e(\omega)]}\, d\omega$$

(A.18)

Considering expression (A.18), we now make the following observations.

$$\lambda_N(\omega) - \lambda_N^e(\omega) \geq 0 \ ; \ \forall \ \omega\varepsilon \ E_\mu$$

$$\mu\nu < 1 \rightarrow \mu^2(\mu+1)^{-2} < (\nu+1)^{-2} \rightarrow \mu^2(\mu+1)^{-1}(1-\varepsilon_N) \ \lambda_{1N}^o(\omega) \ [\lambda_N(\omega) - \lambda_N^e(\omega)] <$$

$$< (\nu+1)^{-2}(\mu+1)(1-\varepsilon_N) \ \lambda_{1N}^o(\omega) \ [\lambda_N(\omega) - \lambda_N^e(\omega)] \ ; \ \forall \ \omega\varepsilon \ E_\mu$$

$$\lambda_s(\omega) - \lambda_s^e(\omega) \geq 0 \ ; \ \forall \ \omega\varepsilon \ E_\nu \ .$$

$$\mu\nu < 1 \rightarrow \nu^2(\nu+1)^{-2} < (\mu+1)^{-2} \rightarrow \nu^2(\nu+1)^{-1}(1-\varepsilon_s) \ \lambda_{1s}^o(\omega) \ [\lambda_s(\omega) - \lambda_s^e(\omega)] <$$

$$< (\mu+1)^{-2}(\nu+1)(1-\varepsilon_s) \ \lambda_{1s}^o(\omega) \ [\lambda_s(\omega) - \lambda_s^e(\omega)] \ ; \ \forall \ \omega\varepsilon \ E_\nu$$

$$\left.\begin{array}{l} \lambda_N(\omega) - \lambda_N^e(\omega) \geq 0 \ , \ \lambda_s(\omega) - \lambda_s^e(\omega) \geq 0 \\ (1-\varepsilon_s)\lambda_{1s}^o(\omega) > \mu(1-\varepsilon_N) \ \lambda_{1N}^o(\omega) \ , \ (1-\varepsilon_N) \ \lambda_{1N}^o(\omega) > \nu(1-\varepsilon_s) \ \lambda_{1s}^o(\omega) \end{array}\right\} ; \ \forall \ \omega\varepsilon \ E_\mu^c \cap E_\nu^c \ \rightarrow$$

$$\text{(A.19)}$$

$$\rightarrow \left\{\begin{array}{l} \dfrac{[(1-\varepsilon_s)\lambda_{1s}^o(\omega)]^2[\lambda_N(\omega)-\lambda_N^e(\omega)]}{(1-\varepsilon_s)\lambda_{1s}^o(\omega) + (1-\varepsilon_N)\lambda_{1N}^o(\omega)} < (\nu+1)^{-1}(1-\varepsilon_s)\lambda_{1s}^o(\omega) \ [\lambda_N(\omega) - \lambda_N^e(\omega)] < \\ \qquad < (\nu+1)^{-2}[(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)][\lambda_N(\omega)-\lambda_N^e(\omega)] \\ \qquad ; \ \forall \ \omega\varepsilon \ E_\mu^c \cap E_\nu^c \\ \\ \dfrac{[(1-\varepsilon_N)\lambda_{1N}^o(\omega)]^2[\lambda_s(\omega)-\lambda_s^e(\omega)]}{(1-\varepsilon_s)\lambda_{1s}^o(\omega) + (1-\varepsilon_N)\lambda_{1N}^o(\omega)} < (\mu+1)^{-1}(1-\varepsilon_N)\lambda_{1N}^o(\omega) \ [\lambda_s(\omega) - \lambda_s^e(\omega)] < \\ \qquad < (\mu+1)^{-2}[(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)][\lambda_s(\omega)-\lambda_s^e(\omega)] \\ \qquad ; \ \forall \ \omega\varepsilon \ E_\mu^c \cap E_\nu^c \end{array}\right.$$

To simplify our notation, we eliminate the integration variable ω, and we denote:

$$\left.\begin{array}{l} x \triangleq \lambda_s-\lambda_s^e, \ y \triangleq \lambda_N-\lambda_N^e, \ z_1 \triangleq (\mu+1)(1-\varepsilon_N)\lambda_{1N}^o \\ z_2 \triangleq (\nu+1)(1-\varepsilon_s)\lambda_{1s}^o, \ z_3 \triangleq (1-\varepsilon_s)\lambda_{1s}^o + (1-\varepsilon_N)\lambda_{1N}^o \end{array}\right\} ; \ \forall \ \omega \qquad \text{(A.20)}$$

In view of the conditions in (A.19) and the notation in (A.20), we obtain from (A.18):

$$\int_{-\pi}^{\pi} \lambda_s \lambda_N \, [\lambda_s + \lambda_N]^{-1} - \int_{-\pi}^{\pi} \lambda_s^e \, \lambda_N^e \, [\lambda_s^e + \lambda_N^e]^{-1} \leq \int_{E_\mu} \frac{xy + (\nu+1)^{-2} \, y \, z_1 + (\mu+1)^{-2} \, x \, z_1}{x + y + z_1} +$$

$$+ \int_{E_\mu^c \cap E_\nu^c} \frac{xy+(\nu+1)^{-2} y \, z_3 + (\mu+1)^{-2} x \, z_3}{x+y+z_3} + \int_{E_\nu} \frac{xy + (\nu+1)^{-2} y \, z_2 + (\mu+1)^{-2} x \, z_2}{x+y+z_2}$$

(A.21)

We observe that all the integrands in the upper bound in (A.21) have the form:

$$f(w_1, w_2, w_3) \triangleq \frac{w_1 \, w_2 + (\nu+1)^{-2} \, w_2 \, w_3 + (\mu+1)^{-2} \, w_1 \, w_3}{w_1 + w_2 + w_3} \tag{A.22}$$

The function $f(w_1, w_2, w_3)$ above is concave in the region determined by $w_1 + w_2 + w_3 > 0$, and $w_3 > 0$; for all ν and μ such that $\mu \, \nu < 1$. Therefore, if Ω is some ω set such that the above conditions for concavity hold, (where w_1, w_2, w_3 functions of ω) an application of Jensen's inequality results in the following expression:

$$\int_\Omega f(w_1, w_2, w_3) \, d\omega \leq f\left(\int_\Omega w_1, \int_\Omega w_2, \int_\Omega w_3\right) \tag{A.23}$$

Considering now the functions defined in (A.20), selecting $\Omega = [-\pi,\pi]$, $w_1 = x$, $w_2 = y$, $w_3 = z_1 \cdot 1_{E_\mu} + z_2 \cdot 1_{E_\nu} + z_3 \cdot 1_{E_\mu^c \cap E_\nu^c}$, and applying (A.23) to (A.22), we obtain:

$$\int_{-\pi}^{\pi} \lambda_s\lambda_N[\lambda_s+\lambda_N]^{-1} - \int_{-\pi}^{\pi} \lambda_s^e\lambda_N^e[\lambda_s^e+\lambda_N^e]^{-1} \leq \frac{\int_{-\pi}^{\pi} x \cdot \int_{-\pi}^{\pi} y + (\nu+1)^{-2} \int_{-\pi}^{\pi} y \cdot \int_{-\pi}^{\pi} w_3 + (\mu+1)^{-2} \int_{-\pi}^{\pi} x \cdot \int_{-\pi}^{\pi} w_3}{\int_{-\pi}^{\pi} x + \int_{-\pi}^{\pi} y + \int_{-\pi}^{\pi} w_3} = 0$$

$$; \text{ since } \int_{-\pi}^{\pi} x = \int_{-\pi}^{\pi} y = 0 \qquad \text{(A.24)}$$

We have now proved that (A.1) and (A.2) are sufficient solutions for the optimization problem (3D). Those solutions clearly satisfy the condition for equality in expression (9); thus they are also sufficient for the supremum of $e_m(\underline{f}_s, \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$, with no restrictions on eigenvectors.

Part 2

Under the conditions (10) in the theorem, the classes F'_{s,ε_s} and F'_{N,ε_N} are non-empty, and $F'_{s,\varepsilon_s} \subset F_{L,\varepsilon_s}$, $F'_{N,\varepsilon_N} \subset F_{L,\varepsilon_N}$. Furthermore, the restrictions through which the classes F'_{s,ε_s} and F'_{N,ε_N} are defined do not imply restrictions on the eigenvectors of the corresponding spectral density matrices. Therefore, for any sets $\{\lambda_{is}(\omega); 1 \leq i \leq n\}$ and $\{\lambda_{iN}(\omega); 1 \leq i \leq n\}$ of ordered eigenfunctions, the eigenvectors of the corresponding spectral density matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$ can be selected arbitrarily for the supremum of $e_m(\underline{f}_s, \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$. Let now $\underline{f}_s(\omega)$ be some spectral density matrix in $F_{L,\varepsilon_s} - F'_{s,\varepsilon_s}$, and let $\underline{f}_N(\omega)$ be some spectral density matrix in $F_{L,\varepsilon_N} - F'_{N,\varepsilon_N}$; where if A and B are two sets, A-B denotes the set with elements in A but not in B. Let $\{\lambda_{is}(\omega);\ 1 \leq i \leq n\}$ and $\{\lambda_{iN}(\omega);\ 1 \leq i \leq n\}$ be the sets of ordered eigenfunctions of the matrices $\underline{f}_s(\omega)$ and $\underline{f}_N(\omega)$ respectively. Then, for every i, there exist $A_i(\omega)$ and $B_i(\omega)$ subsets of $[-\pi,\pi]$, such that $\lambda_{is}(\omega) < (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)$; $\forall\ \omega \epsilon\ A_i(\omega)$ and $\lambda_{iN}(\omega) < (1-\varepsilon_N)\lambda^o_{1N}(\omega)$; $\forall\ \omega \epsilon\ B_i(\omega)$; where $A_n(\omega)$ and $B_n(\omega)$ are necessarily nonempty, and for $i < n$ some of the sets $A_i(\omega)$ and $B_i(\omega)$ may be empty. For any i such that $A_i(\omega)$ is nonempty, we construct a new eigenfunction $\lambda^*_{is}(\omega)$, such that $\lambda^*_{is}(\omega) = \lambda_{is}(\omega)$; $\forall\ \omega \epsilon([-\pi,\pi]-A_i(\omega))$, and

$\lambda^*_{is}(\omega) = (1-\varepsilon_s)\ \lambda^o_{1s}(\omega)$; $\forall\ \omega\varepsilon\ A_i(\omega)$. Similarly, we construct the new eigenfunction $\lambda^*_{iN}(\omega)$, such that $\lambda^*_{iN}(\omega) = \lambda_{iN}(\omega)$; $\forall\ \omega\varepsilon\ ([-\pi,\pi] - B_i(\omega))$ and $\lambda^*_{iN}(\omega) = (1-\varepsilon_N)\ \lambda^o_{1N}(\omega)$; $\forall\ \omega\varepsilon\ B_i(\omega)$. The so constructed sets $\{\lambda^*_{is}(\omega);\ 1\leq i\leq n\}$ and $\{\lambda^*_{iN}(\omega);\ 1\leq i\leq n\}$ correspond to spectral density matrices that are contained in F'_{s,ε_s} and F'_{N,ε_N} respectively. In addition, we have:

$$e_m(\underline{f}_s,\ \underline{f}_N) \leq (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda_{is}(\omega)\right]^{-1} + \left[\sum_{i=1}^{n} \lambda_{iN}(\omega)\right]^{-1} \right)^{-1} d\omega \ <$$

$$< \ (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\left[\sum_{i=1}^{n} \lambda^*_{is}(\omega)\right]^{-1} + \left[\sum_{i=1}^{n} \lambda^*_{iN}(\omega)\right]^{-1} \right)^{-1} d\omega$$

; where the right hand side of the double inequality above is due to the fact that $[x^{-1} + y^{-1}]^{-1}$ increases monotonically with increasing x and y, and due to the fact that the $\{\lambda^*_{is}(\omega);\ 1\leq i\leq n\}$ and $\{\lambda^*_{iN}(\omega);\ 1\leq i\leq n\}$ constructions have this effect as compared to the sets $\{\lambda_{is}(\omega);\ 1\leq i\leq n\}$ and $\{\lambda_{iN}(\omega);\ 1\leq i\leq n\}$.

Thus, it is sufficicent to optimize $e_m(\underline{f}_s,\ \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F'_{N,\varepsilon_N}$.

Proof of Theorem B

As in part 2, in the proof of theorem A, we have again that it is sufficient to optimize $e_m(\underline{f}_s,\ \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F_Q$. Here, we will first show that the solution (B.1) is sufficient for the optimization problem (4A). Then, we will show that this solution is also sufficient for the supremum of $e_m(\underline{f}_s,\ \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F_Q$.

Let $\lambda_s(\omega)$ and $\lambda_N(\omega)$ be as in (A.7). Then, for any admissible sets $\{\lambda_{is}(\omega)\}$, $\{\lambda_{iN}(\omega)\}$, we have that (A.21) holds. Thus, (A.21) also holds if $\{\lambda_{is}(\omega)\}$ and $\{\lambda_{iN}(\omega)\}$ are substituted by $\{\lambda^e_{is}(\omega)\}$ and $\{\lambda^e_{iN}(\omega)\}$ in (B.1). For simplicity in notation, we will denote c'_j as c_j ; for all j, and as in the proof of theorem A, we will prove that:

$$\int_{-\pi}^{\pi} [\lambda_s^{-1}(\omega) + \lambda_N^{-1}(\omega)]^{-1}\, d\omega \le \int_{-\pi}^{\pi} \lambda_s^{e}(\omega) \quad \lambda_N^{e}(\omega)[\lambda_s^{e}(\omega) + \lambda_N^{e}(\omega)]^{-1}\, d\omega \tag{A.25}$$

Indeed, from the definition of $\lambda_s^e(\omega)$ and $\lambda_N^e(\omega)$ in (B.1), we easily obtain:

$$\int_{-\pi}^{\pi} [\lambda_s^{-1}(\omega) + \lambda_N^{-1}(\omega)]^{-1}\, d\omega - \int_{-\pi}^{\pi} \lambda_s^{e}(\omega) \quad \lambda_N^{e}(\omega)[\lambda_s^{e}(\omega) + \lambda_N^{e}(\omega)]^{-1}\, d\omega \cdot$$

$$= \sum_{j=0}^{k} \left(\int_{A_j} \frac{\lambda_s(\omega)\ \lambda_N(\omega)}{\lambda_s(\omega)+\lambda_N(\omega)}\, d\omega - n^{-1} \frac{c_j\ x_j}{c_j+x_j} \right) \tag{A.26}$$

Applying the lemma to expression (A.26), we obtain.

$$\int_{-\pi}^{\pi} [\lambda_s^{-1}(\omega) + \lambda_N^{-1}(\omega)]^{-1}\, d\omega - \int_{-\pi}^{\pi} \lambda_s^{e}(\omega)\ \lambda_N^{e}(\omega)[\lambda_s^{e}(\omega) + \lambda_N^{e}(\omega)]^{-1} d\omega \le$$

$$\le \sum_{j=0}^{k} \left(\frac{\int_{A_j} \lambda_s(\omega)\, d\omega \cdot \int_{A_j} \lambda_N(\omega)\ d\omega}{\int_{A_j} \lambda_s(\omega)\, d\omega + \int_{A_j} \lambda_N(\omega)\ d\omega} - n^{-1} \frac{c_j\ x_j}{c_j+x_j} \right) =$$

$$= n^{-1} \sum_{j=0}^{k} \left(\frac{c_j\ y_j}{c_j+y_j} - \frac{c_j\ x_j}{c_j+x_j} \right) = n^{-1} \sum_{j=0}^{k} \frac{c_j^2\ (y_j - x_j)}{(c_j+x_j)(c_j+y_j)} =$$

$$= n^{-1} \sum_{j:\mu c_j \ge x_j^o} \frac{c_j^2\ (y_j - x_j)}{(c_j+\mu c_j)(c_j+y_j-x_j+\mu c_j)} + n^{-1} \sum_{j:\mu c_j < x_j^o} \frac{c_j^2\ (y_j - x_j)}{(c_j+x_j^o)(c_j+y_j-x_j+x_j^o)} \tag{A.27}$$

$$; \text{ where } \qquad y_j \triangleq \int_{A_j} \lambda_s(\omega)\, d\omega \tag{A.28}$$

Since $\lambda_s(\omega) \geq (1-\varepsilon_s)\ \lambda_{1s}^o(\omega) \rightarrow$

$$\rightarrow \begin{cases} y_j \geq x_j^o \quad , \quad y_j - x_j \geq 0 \quad ; \quad j: \mu\ c_j < x_j^o \\ \sum_{j=0}^{k} y_j = \sum_{j=0}^{k} x_j = n^{-1}\ W_1 \end{cases} \tag{A.29}$$

Now, the expression in (A.27) can be bounded from above by

$$n^{-1} \sum_{j=0}^{k} (\mu+1)^{-2}\ \frac{(\mu+1)c_j(y_j-x_j)}{(\mu+1)c_j+(y_j-x_j)}$$

which, due to corollary 1 in the paper, is bounded from above by:

$$n^{-1}(\mu+1)^{-2}\ \frac{\left[\sum_{j=0}^{k} (\mu+1)c_j\right]\left[\sum_{j=0}^{k} (y_j-x_j)\right]}{\sum_{j=0}^{k} (\mu+1)c_j + \sum_{j=0}^{k} (y_j-x_j)} \tag{A.30}$$

Due to (A.29), the expression in (A.30) equals zero, and (A.25) is now proved. That proves that (B.1) is a sufficient solution for the optimization problem (4A). But, this solution clearly satisfies the conditions for equality in (9). Thus, it is also sufficient for the optimization of $e_m(\underline{f}_s, \underline{f}_N)$ on $F'_{s,\varepsilon_s} \times F_Q$.

Proof of Theorem C

As in the proof of theorem B, we will prove that (A.25) holds, for the $\lambda_s^e(\omega)$, $\lambda_N^e(\omega)$ selections as in (C.1), and for all admissible $\{\lambda_s(\omega),\ \lambda_N(\omega)\}$ pairs.

Since,

$$A_{si} \cap A_{sj} = \emptyset\ ;\ \forall\ i \neq j\ ,\ \bigcup_{j=1}^{k} A_{sj} = [-\pi,\pi]$$

$$A_{Ni} \cap A_{Nj} = \emptyset\ ;\ \forall\ i \neq j\ ,\ \bigcup_{j=1}^{m} A_{Nj} = [-\pi,\pi]$$

it is clear that the km sets $A_{si} \cap A_{Nj}$; i=1,...,k,j=1,...,m are mutually exclusive and exhaustive. Thus,

$$\int_{-\pi}^{\pi} \lambda_s(\omega)\ \lambda_N(\omega)[\lambda_s(\omega) + \lambda_N(\omega)]^{-1}\, d\omega = \sum_{j=1}^{k} \int_{A_{sj}} \lambda_s(\omega)\ \lambda_N(\omega)[\lambda_s(\omega) + \lambda_N(\omega)]^{-1}\, d\omega$$

$$\leq \sum_{j=1}^{k} \frac{\int_{A_{sj}} \lambda_s(\omega)\,d\omega + \int_{A_{sj}} \lambda_N(\omega)\,d\omega}{\int_{A_{sj}} \lambda_s(\omega)\,d\omega + \int_{A_{sj}} \lambda_N(\omega)\,d\omega} = \sum_{j=1}^{k} \frac{n^{-1}\ c_{sj} \sum_{\ell=1}^{m} \int_{A_{sj} \cap A_{N\ell}} \lambda_N(\omega)\,d\omega}{n^{-1}\ c_{sj} + \sum_{\ell=1}^{m} \int_{A_{sj} \cap A_{N\ell}} \lambda_N(\omega)\,d\omega}$$

$$= \int_{-\pi}^{\pi} \lambda_s^e(\omega)\ \lambda_N^e(\omega)[\lambda_s^e(\omega) + \lambda_N^e(\omega)]^{-1}\, d\omega \qquad \text{(A.31)}$$

; where the inequality in (A.31) is due to the lemma.

Thus, the inequality (A.25) holds in this case, and the proof is complete, for the problem (5A). The solution satisfies the condition for equality in (9), thus it is sufficient for the optimization of $e_m(\underline{f}_s, \underline{f}_N)$ on $F_{Q_s} \times F_{Q_N}$.

<u>Proof of Theorem 1</u>

a.

Expression (26) is easily derived, by substitution of solution (A.1) in theorem A, in (21).

Let now condition (12) be satisfied. Then, directly from solution (A.2) in theorem A, and from the proven (in the proof of theorem A) fact that $\mu\ \nu < 1$, we have:

$$\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \begin{cases} [1+\mu]^{-1} & ;\ \omega : \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \le \mu \\ \nu[1+\nu]^{-1} & ;\ \omega : \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \ge \nu^{-1} \\ \dfrac{(1-\varepsilon_N)\lambda_{1N}^o(\omega)}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)} & ;\ \omega : \mu < \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} < \nu^{-1} \end{cases} \quad \text{(A.32)}$$

$$\frac{\lambda_s^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \begin{cases} \mu(1+\mu)^{-1} & ;\ \omega : \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \le \mu \\ [1+\nu]^{-1} & ;\ \omega : \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \ge \nu^{-1} \\ \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)} & ;\ \omega : \mu < \dfrac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} < \nu^{-1} \end{cases} \quad \text{(A.33)}$$

Now, due to $\mu\ \nu < 1$, we have:

$$\nu[1+\nu]^{-1} < \min\left([1+\mu]^{-1}\ ,\ \frac{(1-\varepsilon_N)\lambda_{1N}^o(\omega)}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)}\right) \quad ;$$

$$;\ \omega : \mu < \frac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} < \nu^{-1} \quad \text{(A.34)}$$

$$\mu[1+\mu]^{-1} < \min\left([1+\nu]^{-1}\ ,\ \frac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_s)\lambda_{1s}^o(\omega)+(1-\varepsilon_N)\lambda_{1N}^o(\omega)}\right) \quad ;$$

$$;\ \omega : \mu < \frac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o\ \omega} < \nu^{-1} \quad \text{(A.35)}$$

From (A.32), (A.33), (A.34), and (A.35), we clearly obtain.

$$\min_{\omega}\left(\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)}\right) = \nu[1+\nu]^{-1}; \text{ achieved at all } \omega: \frac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \geq \nu^{-1} \quad (A.36)$$

$$\min_{\omega}\left(\frac{\lambda_s^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)}\right) = \mu[1+\mu]^{-1}; \text{ achieved at all } \omega: \frac{(1-\varepsilon_s)\lambda_{1s}^o(\omega)}{(1-\varepsilon_N)\lambda_{1N}^o(\omega)} \leq \mu \quad (A.37)$$

Due to the fact that the infimum in (21) is a linear programming problem, and due to (A.36) and (A.37), we easily obtain by substitution the result in (27).

b.

By substitution of the solution (B.1) in theorem B, we obtain:

$$\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \sum_{j=0}^{k} \frac{c_j}{c_j+x_j} 1_{A_j}(\omega) \quad (A.38)$$

$$\frac{\lambda_s^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \sum_{j=0}^{k} \frac{x_j}{c_j+x_j} 1_{A_j}(\omega) \quad (A.39)$$

From the definition of the x_j's in theorem B, and from (A.38), we obtain:

$$\min_{\omega}\left(\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)}\right) = \left[1 + \max_{j:x_j^o>\mu c_j} (x_j^o c_j^{-1})\right]^{-1} \quad (A.40)$$

Finally, due to the fact that the infimum in (21) is a linear programming problem, and due to (A.40), we obtain the result in (28).

c.

Without lack in generality, we will consider the special solution (C.2), in

theorem C. We easily find then, by substitution:

$$\frac{\lambda_N^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \left[1 + \frac{c_{sj}}{\sum_{\ell=1}^{m} c_{N\ell}\, \mu(A_{n\ell} \cap A_{sj})\mu^{-1}(A_{N\ell})}\right]^{-1}; \ \omega \in A_{sj} \tag{A.41}$$

$$\frac{\lambda_s^e(\omega)}{\lambda_s^e(\omega)+\lambda_N^e(\omega)} = \left[1 + c_{sj}^{-1} \sum_{\ell=1}^{m} c_{N\ell}\, \mu(A_{N\ell} \cap A_{sj})\ \mu^{-1}(A_{N\ell})\right]^{-1} \ ; \ \omega \in A_{sj} \tag{A.42}$$

From (A.41), (A.42), and (21), we easily find the expression (29), by direct substitution.

Proof of Theorem 2

Let us first consider the conditions (10) in theorem A, where W_1 and W_2 are given by (3A). From those conditions, we directly obtain.

$$\begin{aligned} \varepsilon_s &\geq \left\{1 + 2\pi\ W_s\left[\int_{-\pi}^{\pi} n\ \lambda_{1s}^o(\omega)\ d\omega - 2\pi\ W_{os}\right]^{-1}\right\}^{-1} \\ \varepsilon_N &\geq \left\{1 + 2\pi\ W_N\left[\int_{-\pi}^{\pi} n\ \lambda_{1N}^o(\omega)\ d\omega - 2\pi\ W_{oN}\right]^{-1}\right\}^{-1} \end{aligned} \tag{A.43}$$

Let us now define,

$$\begin{aligned} y &\triangleq (2\pi)^{-1}\ [W_{os} + \varepsilon_s[1-\varepsilon_s]^{-1}\ W_s] \\ x &\triangleq (2\pi)^{-1}\ [W_{oN} + \varepsilon_N(1-\varepsilon_N]^{-1}\ W_N] \\ f_1(\omega) &\triangleq n\ \lambda_{1s}^o(\omega) \\ f_2(\omega) &\triangleq n\ \lambda_{1N}^o(\omega) \\ F(y,x) &\triangleq \int_{-\pi}^{\pi} \max\ (y\ f_1(\omega),\ x\ f_2(\omega))\ d\omega \end{aligned} \tag{A.44}$$

Then, the quantities y, x defined above are monotonically increasing with ε_s, ε_N respectively, the conditions (A.43) in terms of y and x become,

$$y \geq (2\pi)^{-2} \int_{-\pi}^{\pi} n\lambda^{o}_{1s}(\omega)\ d\omega$$
$$x \geq (2\pi)^{-2} \int_{-\pi}^{\pi} n\lambda^{o}_{1N}(\omega)\ d\omega \tag{A.45}$$

and the acceptable $(\varepsilon_s, \varepsilon_N)$ region corresponds to,

$$(y, x) : F(y, x) \geq 1 \tag{A.46}$$

From the expression of F(y, x) in (A.44), it is clear that for the satisfaction of (A.46), it is necessary that:

$$\max_{\omega} \max \left(1, \frac{x\ f_2(\omega)}{y\ f_1(\omega)}\right) \cdot y \int_{-\pi}^{\pi} f_1(\omega)\ d\omega \geq 1 \tag{A.47}$$

; where, due to conditions (10):

$$y \int_{-\pi}^{\pi} f_1(\omega)\ d\omega \leq 1 \tag{A.48}$$

From (A.47) and (A.48) we conclude that for the satisfaction of (A.46) it is necessary that:

$$x \geq \left[\min_{\omega} \left(\frac{f_1(\omega)}{f_2(\omega)}\right)\right]\left[\int_{-\pi}^{\pi} f_1(\omega)\ d\omega\right]^{-1}$$
$$y \geq \left[\min_{\omega} \left(\frac{f_2(\omega)}{f_1(\omega)}\right)\right]\left[\int_{-\pi}^{\pi} f_2(\omega)\ d\omega\right]^{-1} \tag{A.49}$$

For the satisfaction of both the conditions (A.45) and (A.49), we finally require:

$$x \geq \left[\int_{-\pi}^{\pi} n\, \lambda_{1s}^{o}(\omega)\, d\omega\right]^{-1} \cdot \max\left(\min_{\omega} g(\omega),\ (2\pi)^{-2}\, D\left[\int_{-\pi}^{\pi} n\, \lambda_{1s}^{o}(\omega)\, d\omega\right]^{2}\right)$$

$$y \geq \left[\int_{-\pi}^{\pi} n\, \lambda_{1N}^{o}(\omega)\, d\omega\right]^{-1} \cdot \max\left(\min_{\omega} g^{-1}(\omega),\ (2\pi)^{-2}\, D\left[\int_{-\pi}^{\pi} n\, \lambda_{1s}^{o}(\omega)\, d\omega\right]^{2}\right) \qquad \text{(A.50)}$$

; where $g(\omega)$ and D are given by (38). The regions $[B_N, 1]$ and $[B_s, 1]$ are obtained by substitution of the y and x in (A.44), in the conditions (A.50).

Let us now define,

$$G(z) \triangleq \int_{-\pi}^{\pi} \max\left(z\, f_1(\omega),\ f_2(\omega)\right) d\omega \qquad \text{(A.51)}$$

Then, we can trivially write,

$$F(y, x) = x\, G\left(\frac{y}{x}\right) \qquad \text{(A.52)}$$

We will define $z = \frac{y}{x}$, and we will use z as a parameter. The breakdown curve is determined by the equation

$$F(y, x) = 1$$

In a parametrized form, this equation becomes,

$$\begin{aligned} x(z) &= G^{-1}(z) \\ &\text{with} \\ y(z) &= z\, G^{-1}(z) \end{aligned} \qquad \text{(A.53)}$$

Let us first study the function $G(z)$; $0 \leq z < \infty$. We first define,

$$E_z \triangleq \left\{ \omega : z \geq \frac{f_2(\omega)}{f_1(\omega)} \right\} \tag{A.54}$$

and clearly,

$$E_{z+dz} \supseteq E_z \; ; \forall \; z, \; dz > 0 \tag{A.55}$$

Due to the conditions satisfied by the function $g(\omega) = f_1(\omega) \, f_2^{-1}(\omega)$, we have that,

$$\lim_{dz \to 0} E_{z+dz} = E_z \; ; \forall \; z \geq 0 \tag{A.56}$$

Also, we can write:

$$G(z) = \int_{E_z} \left[z \, f_1(\omega) - f_2(\omega) \quad d\omega \right] + \int_{-\pi}^{\pi} f_2(\omega) \; d\omega \tag{A.57}$$

From (A.57), and due to (A.55) and (A.56), we easily find,

$$\begin{aligned} &\frac{\partial}{\partial z} G(z) \triangleq \lim_{dz \to 0} \frac{G(z+dz)-G(z)}{dz} = \int_{E_z} f_1(\omega) \; d\omega \geq 0 \; ; \forall \; z \geq 0 \\ &\frac{\partial^2}{\partial z^2} G(z) \triangleq \lim_{dz \to 0} \frac{\int_{E_{z+dz}} f_1(\omega) d \; - \int_{E_z} f_1(\omega) \; d\omega}{dz} \geq 0 \end{aligned} \tag{A.58}$$

Now, from (A.53) we find in a straightforward fashion:

$$\begin{aligned} &\frac{dy(z)}{dx(z)} = - \, z - G(z) \left[\frac{\partial}{\partial z} \; G(z) \right]^{-1} \\ &\frac{d^2 y(z)}{d^2 x(z)} = z - \frac{2 \; G(z) \left[\frac{\partial}{\partial z} G(z) \right]}{2 \left[\frac{\partial}{\partial z} G(z) \right]^2 - G(z) \left[\frac{\partial^2}{\partial z^2} G(z) \right]} \end{aligned} \tag{A.59}$$

Due to expressions (A.58), we finally obtain:

$$\frac{dy(z)}{dx(z)} = -\left[\int_{E_z} f_1(\omega)\, d\omega\right]^{-1} \left[\int_{E_z^c} f_2(\omega)\, d\omega\right] \tag{A.60}$$

$$\frac{d^2y(z)}{d^2x(z)} < 0 \; ; \; \forall\, z > 0 \tag{A.61}$$

Due to (A.60) and (A.61) above, we conclude that in terms of y and x variables, the breakdown curve is monotone and concave and that (considering also (A.54)):

$$\left.\frac{dy(z)}{dx(z)}\right|_{z = 0} \to -\infty$$

$$\left.\frac{dy(z)}{dx(z)}\right|_{z \to \infty} = 0 \tag{A.62}$$

But $z = 0$ corresponds to $\varepsilon_N = 1$, and $z \to \infty$ corresponds to $\varepsilon_s = 1$. Also, y and x are monotonic with respect to ε_s and ε_N respectively. Thus, the conclusions in the theorem.

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